The Field of Algebraic Numbers

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Warm-up: The Additive Inverse of an Algebraic Number is Algebraic.

```
P[x_] = x^7 - 4 x^6 + 2 x^5 - 2 x^3 + 3 x^2 - x + 3
```

$$x^7 - 4\,x^6 + 2\,x^5 - 2\,x^3 + 3\,x^2 - x + 3$$

```
Sort[NSolve[P[x] == 0][[All, 1, 2]],
 If[Re[#1] == Re[#2], Im[#1] < Im[#2], Re[#1] < Re[#2]] &]
```

$$\{-1.05898,\ -0.242683 - 0.77271\,i,\ -0.242683 + 0.77271\,i,\ 0.467258 - 0.927124\,i,\ 0.467258 + 0.927124\,i,\ 1.16208,\ 3.44775\}$$

```
Q[x_] = -x^7 - 4 x^6 - 2 x^5 + 2 x^3 + 3 x^2 + x + 3
```

$$-x^7 - 4\,x^6 - 2\,x^5 + 2\,x^3 + 3\,x^2 + x + 3$$

```
Sort[NSolve[Q[x] == 0][[All, 1, 2]],
 If[Re[#1] == Re[#2], Im[#1] > Im[#2], Re[#1] > Re[#2]] &]
```

$$\{1.05898,\ 0.242683 + 0.77271\,i,\ 0.242683 - 0.77271\,i,\ -0.467258 + 0.927124\,i,\ -0.467258 - 0.927124\,i,\ -1.16208,\ -3.44775\}$$

The Resultant

In[2]:=
```
Resultant[(x - a) (x - b), (x - c) (x - d) (x - e), x]
```

Out[2]=

$$(c - a)\,(d - a)\,(e - a)\,(-(b - c))\,(b - d)\,(b - e)$$

In[3]:=
```
P[x_] = x^4 - 5
Q[x_] = x^4 - 2 x^2 - 5
```

Out[3]= $x^4 - 5$

Out[4]= $x^4 - 2\,x^2 - 5$

In[5]:=
```
Solve[P[x] == 0]
```

Out[5]= $\left\{\left\{x \to -\sqrt[4]{5}\right\}, \left\{x \to -i\,\sqrt[4]{5}\right\}, \left\{x \to i\,\sqrt[4]{5}\right\}, \left\{x \to \sqrt[4]{5}\right\}\right\}$

In[6]:=
```
Solve[Q[x] == 0]
```

Out[6]= $\left\{\left\{x \to -i\,\sqrt{\sqrt{6}-1}\right\}, \left\{x \to i\,\sqrt{\sqrt{6}-1}\right\}, \left\{x \to -\sqrt{1+\sqrt{6}}\right\}, \left\{x \to \sqrt{1+\sqrt{6}}\right\}\right\}$

In[7]:=
```
Resultant[P[x], Q[x], x]
```

Out[7]= 400

The Sum of Two Algebraic Numbers is Algebraic.

```
Clear[R]
R[z_] = Resultant[P[x], Q[z - x], x]
```

$z^{16} - 8\,z^{14} - 16\,z^{12} + 48\,z^{10} - 3304\,z^8 + 5280\,z^6 - 36\,160\,z^4 + 1600\,z^2 + 400$

```
NSolve[R[z] == 0]
```

$\{\{z \to -3.35263\}, \{z \to -1.85728 - 1.49535\,i\}, \{z \to -1.85728 + 1.49535\,i\},$
$\{z \to -1.49535 - 1.20395\,i\}, \{z \to -1.49535 + 1.20395\,i\}, \{z \to -0.361931\}, \{z \to 0. - 0.291401\,i\},$
$\{z \to 0. + 0.291401\,i\}, \{z \to 0. - 2.6993\,i\}, \{z \to 0. + 2.6993\,i\}, \{z \to 0.361931\}, \{z \to 1.49535 - 1.20395\,i\},$
$\{z \to 1.49535 + 1.20395\,i\}, \{z \to 1.85728 - 1.49535\,i\}, \{z \to 1.85728 + 1.49535\,i\}, \{z \to 3.35263\}\}$

```
- 5.^.25 + -i Sqrt[Sqrt[6] - 1]
```

$-1.49535 - 1.20395\,i$

The Product of Two Algebraic Numbers is Algebraic.

```
Resultant[P[x], x^2 Q[t / x], x]
```

$$\frac{1}{25}\left(t^8 - 70\,t^4 + 625\right)^2$$

```
Solve[% == 0]
```

$$\left\{\left\{t \to -\sqrt[4]{5\left(7-2\sqrt{6}\right)}\right\}, \left\{t \to -\sqrt[4]{5\left(7-2\sqrt{6}\right)}\right\}, \left\{t \to -i\sqrt[4]{5\left(7-2\sqrt{6}\right)}\right\}, \left\{t \to -i\sqrt[4]{5\left(7-2\sqrt{6}\right)}\right\},\right.$$
$$\left\{t \to i\sqrt[4]{5\left(7-2\sqrt{6}\right)}\right\}, \left\{t \to i\sqrt[4]{5\left(7-2\sqrt{6}\right)}\right\}, \left\{t \to \sqrt[4]{5\left(7-2\sqrt{6}\right)}\right\}, \left\{t \to \sqrt[4]{5\left(7-2\sqrt{6}\right)}\right\},$$
$$\left\{t \to -\sqrt[4]{5\left(7+2\sqrt{6}\right)}\right\}, \left\{t \to -\sqrt[4]{5\left(7+2\sqrt{6}\right)}\right\}, \left\{t \to -i\sqrt[4]{5\left(7+2\sqrt{6}\right)}\right\}, \left\{t \to -i\sqrt[4]{5\left(7+2\sqrt{6}\right)}\right\},$$
$$\left.\left\{t \to i\sqrt[4]{5\left(7+2\sqrt{6}\right)}\right\}, \left\{t \to i\sqrt[4]{5\left(7+2\sqrt{6}\right)}\right\}, \left\{t \to \sqrt[4]{5\left(7+2\sqrt{6}\right)}\right\}, \left\{t \to \sqrt[4]{5\left(7+2\sqrt{6}\right)}\right\}\right\}$$

```
N[%]
```

{{$t \to -1.80032$}, {$t \to -1.80032$}, {$t \to 0. - 1.80032\,i$}, {$t \to 0. - 1.80032\,i$}, {$t \to 0. + 1.80032\,i$},
{$t \to 0. + 1.80032\,i$}, {$t \to 1.80032$}, {$t \to 1.80032$}, {$t \to -2.77728$}, {$t \to -2.77728$}, {$t \to 0. - 2.77728\,i$},
{$t \to 0. - 2.77728\,i$}, {$t \to 0. + 2.77728\,i$}, {$t \to 0. + 2.77728\,i$}, {$t \to 2.77728$}, {$t \to 2.77728$}}

```
(- 5.^.25) (-i Sqrt[Sqrt[6] - 1])
```

$0. + 1.80032\,i$

The Multiplicative Inverse of an Algebraic Numbers is Algebraic.

```
Solve[y^4 Q[1 / y] == 0]
```

$$\left\{\left\{y \to -\sqrt{\frac{1}{5}\left(\sqrt{6}-1\right)}\right\}, \left\{y \to \sqrt{\frac{1}{5}\left(\sqrt{6}-1\right)}\right\}, \left\{y \to -i\sqrt{\frac{1}{5}\left(1+\sqrt{6}\right)}\right\}, \left\{y \to i\sqrt{\frac{1}{5}\left(1+\sqrt{6}\right)}\right\}\right\}$$

```
N[%]
```

$\{\{y \to -0.538422\}, \{y \to 0.538422\}, \{y \to 0. - 0.830601\,i\}, \{y \to 0. + 0.830601\,i\}\}$

```
1/(-i Sqrt[Sqrt[6.] - 1])
```

$0. + 0.830601\,i$