Solids in $\mathbb{R}^n$ whose Area is the Derivative of the Volume

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1 Introduction

One of Archimedes' favorite results was the fact that both the volume and surface area ratios of a sphere to its circumscribed cylinder are $\frac{2}{3}$. After studying Archimedes in our department's Great Theorems in Mathematics class, the second author assigned the following homework problem: "Show that, for the bicylinder (the solid formed by the orthogonal intersection of two congruent right circular cylinders) and its circumscribing cube, the volume and surface area ratios are also $\frac{2}{3}$." This problem has been discussed by DeTemple [2], and the most difficult part is finding the surface area of the bicylinder. However, some of the students sidestepped the difficulty. They first found the volume in terms of the radius, r, of the intersecting cylinders, using the usual calculus I method, and then claimed that the surface area is simply the derivative of the volume with respect to r. Surprisingly, this procedure does give the correct surface area, and the ratios of $\frac{2}{3}$ are obtained.

We were intrigued by the students' work, and this paper is the result of our attempt to answer the question, "When is surface area equal to the derivative of volume?" The only other examples we knew of, at first, were the circle in $\mathbb{R}^2$, where $\frac{d}{dr}\left(\pi r^2\right) = 2\pi r$, and the sphere in $\mathbb{R}^3$, where $\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$.

2 Regions in $\mathbb{R}^2$

In the $\mathbb{R}^2$ case, we want to explore the equation

$$\frac{d\,A}{d\,r} = P, \tag{1}$$

where A is area, P is perimeter, and r is some linear dimension. In [7] Tong showed that (1) will always hold if $r = 2A/P$. However, we are interested in a geometric understanding of this idea, especially in cases where r has additional geometric significance. Consider the case of a square with side s, so that $A = s^2$ and $P = 4s$. With $r = s$, (1) does not hold, and Tong's condition says we should use $r = s/2$ instead. Then, $A = 4r^2$, $P = 8r$, and (1) holds. Here, the r that "works" is the radius of the inscribed circle, and we can generalize this idea to see that (1) holds for convex figures whose boundary consists of straight sides tangent to the circle and arcs of the circle. Although we need the figure to have an inscribed circle, the key idea is that by using the center of the circle, we can decompose such convex figures into triangles and sectors which have the desired relationship between the derivatives of the area and the length of the exterior side.

Lemma 1. *Let R be either the region formed by a sector of a circle of radius r or a triangle with altitude of length r. Also, let ∂R be the arc of the sector or the side perpendicular to the altitude r, and L be the length of ∂R. Then R satisfies*

$$\frac{d\,A}{d\,r} = L.$$

Proof. In the case of the triangular region, the length of ∂R is linearly proportional to the length of the altitude, r. Hence $L = kr$, for some constant k and $A = \frac{1}{2}kr^2$ (see FIGURE 1). For the region formed by a sector, let θ be the central angle of the sector. Then $L = r\theta$ and $A = \frac{1}{2}r^2\theta$ (see FIGURE 2). Thus, in both cases the result holds. □

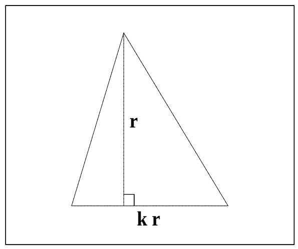

FIGURE 1: Triangular region.

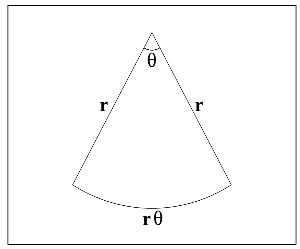

FIGURE 2: Sectorial region.

Therefore, whenever a region is finitely decomposible into triangular and sectorial regions in this manner, we will get eq. (1) to hold.

Theorem 2. *Any convex region R in $\mathbb{R}^2$ having an inscribed circle S^1 of radius r such that every straight side of the boundary of R is tangent to S^1 and every curved arc of the boundary of R is an arc of S^1 has the property*

$$\frac{d\,A}{d\,r} = P.$$

In particular, using the radius of the inscribed circle as the variable of differentiation, the result holds for triangles, rhombuses, and all regular polygons.[1]

Exercise 1. Show that $\frac{dA}{dr} = P$ for a $30^\circ - 60^\circ - 90^\circ$ triangle with an inscribed circle of radius r.[2]

If the circle S^1 does not touch each side of the convex region R, then the result as stated in eq. (1) does not hold. For example, if R is a rectangle of length $4r$ and height $2r$, a circle of radius r can be tangent to at most three sides of the rectangle, and $dA/dr \neq P$. However, we can modify the result in the following way. For any point O in the interior of the rectangle, let r_1, r_2, r_3, and r_4 be the distances from O to the sides. We can express the area and the perimeter in terms of r_1, r_2, r_3, and r_4 by

$$A = (r_1 + r_3)(r_2 + r_4)$$

and

$$P = 2(r_1 + r_3) + 2(r_2 + r_4).$$

[1] Miller [6] showed this for regular polygons.

[2] Upon request, the authors will provide solutions to any of the exercises.

Now,

$$P = \sum_{i=1}^{4} \frac{\partial A}{\partial r_i}.$$

We can generalize this process for certain regions that are starlike with respect to a point in their interior (a region R is starlike with respect to a point O if the line segment connecting O to any point p on ∂R lies entirely in the interior of R).

Theorem 3. *Let R be a region in $\mathbb{R}^2$ that is starlike with respect to some point O in the interior of R and whose boundary consists of arcs of circles whose center is at O and of straight sides. Then R satisfies*

$$\sum_{i=1}^{n} \frac{\partial\, A}{\partial\, r_i} = P,$$

where r_i is the length of the radius of a sector or is the length of the altitude from O to a straight side.

In some cases, the r_i's in Theorem 3 can be related to Tong's value of r, giving his r an additional geometric meaning.

Theorem 4. *Let R be a n-sided polygon that is starlike with respect to a point T in the interior of R such that R can be triangulated into n triangles of equal area with bases the sides of the polygon and opposite vertex T. If r_i is the distance from T to the ith side of R and r is the harmonic mean of $r_1, \ldots, r_n$, then*

$$\frac{d\, A}{d\, r} = P.$$

Proof. As in the proof of Lemma 1, each side, s_i, of R is proportional to the corresponding r_i, so

$$A = \sum_{i=1}^{n} \frac{1}{2} s_i r_i = \sum_{i=1}^{n} \frac{1}{2} k_i r_i^2$$

$$P = \sum_{i=1}^{n} k_i r_i = \sum_{i=1}^{n} \frac{1}{r_i} k_i r_i^2.$$

Because each triangle has the same area, we have

$$A = \frac{n}{2} k_1 r_1^2 \qquad \text{and} \qquad P = k_1 r_1^2 \sum_{i=1}^{n} \frac{1}{r_i},$$

and the r for which $dA/dr = P$ is

$$r = \frac{2A}{P} = \frac{n}{\displaystyle\sum_{i=1}^{n} \frac{1}{r_i}},$$

which is the harmonic mean of $r_1, \ldots, r_n$. □

When R is a triangle, its centroid satisfies the conditions for T and

$$r = \frac{3 r_1 r_2 r_3}{r_2 r_3 + r_1 r_3 + r_1 r_2}.$$

This provides another proof of the following known result [5].

Corollary 5. *For any triangle, the harmonic mean of its altitudes is three times the inradius of the triangle.*

The equal-area triangulation point T does not exist for every polygon, although it clearly does exist for any triangle and any regular polygon. For a quadrilateral, T will exist if and only if one of the diagonals bisects the quadrilateral into two triangles of equal area. In this case, T is the midpoint of the bisecting diagonal and may or may not be the centroid. For a parallelogram, T is the intersection of the diagonals and does coincide with the centroid.It is straightforward to generalize Theorem 4.

Theorem 6. *Let T be any point in the interior of a $n-$gon such that the n-gon is starlike with respect to T and triangulate the polygon from T. Let the triangle with side s_i and altitude r_i from s_i to T have area A_i. Then $dA/dr = P$ for the polygon, where*

$$\frac{1}{r} = \sum_{i=1}^{n} \frac{A_i}{A} \frac{1}{r_i}.$$

Besides thinking of the regions that satisfy the conditions of Theorem 3 as a decomposition into triangular and sectorial regions whose "bases" form a connected Jordan curve representing the boundary of R and which share the common vertex point O, these regions R can be viewed as the union or intersection of regions that satisfy the conditions of Theorem 2.

3 Solids in $\mathbb{R}^3$

In the same way that a circle of radius r plays an essential role for regions in $\mathbb{R}^2$, a sphere of radius r is crucial for understanding when the equation

$$\frac{d\,V}{d\,r} = A \tag{2}$$

holds for solids, S, in $\mathbb{R}^3$, where V is volume and A is surface area. Before we establish that the results from the previous section have analogues for solids, we will explore a relationship between a circle and a sphere. This allows us to prove ways in which a solid in $\mathbb{R}^3$ with the desired property can be created from a region in $\mathbb{R}^2$ satisfying eq. (1). We can think of a sphere as the revolution of a circle about a diameter line. Hence, we will consider surfaces that are formed by revolving regions from $\mathbb{R}^2$ for which eq. (1) holds.

Theorem 7. *Let $R \in \mathbb{R}^2$ be a region which satisfies the conditions of Theorem 2 and which is symmetric with respect to an axis through the center O of S^1. Let $S \in \mathbb{R}^3$ be the solid formed by revolving R about that axis of symmetry. Then*

$$\frac{d\,V}{d\,r} = A.$$

Proof. Finding the volume and the surface area in this case is simplifed by using Pappus' Centroid Theorems (see [3], pp. 915-917). These theorems are:

- The volume of a solid of revolution generated by revolving a planar region R about an axis that does not intersect the interior of R is

  $$V = (\text{Area of } R) \cdot d_\alpha = 2\pi \overline{x}_\alpha \cdot (\text{Area of } R),$$

  where d_α is the distance traveled by the area's centroid $\overline{x}_\alpha$.

- Let a plane curve γ lie on one side of an axis in the plane. The surface area A of the surface of revolution generated by revolving γ about that axis is

$$A = s \cdot d_\gamma = 2\pi \overline{x}_\gamma \cdot s,$$

where s is the arc length of the curve and d_γ is the distance traveled by the curve's centroid $\overline{x}_\gamma$.

Since the region R satisfying Theorem 2 can be separated into triangular and sectorial regions, it suffices to show that the result holds for these regions with sides above or on the x-axis and a vertex at the origin. First, consider the case of the triangular region. Let θ_1, θ_2 be the angles formed from the sides and the median emanating from the vertex at the origin (see FIGURE 3). The centroid $\overline{x}_\gamma$ of the other side is the midpoint of that side and the centroid of the area is the intersection of the medians of the triangle. Hence, $\overline{x}_\alpha = \frac{2}{3}\overline{x}_\gamma$, and because both are linear dimensions, $\overline{x}_\gamma = kr$, where r is the length of the altitude from the origin. Thus, we have

$$A = 2\pi(k\,r)(r\tan\theta_1 + r\tan\theta_2),$$

while

$$V = 2\pi(\tfrac{2}{3}k\,r)(\tfrac{1}{2}r^2\tan\theta_1 + \tfrac{1}{2}r^2\tan\theta_2).$$

Hence, eq. (2) holds.

Next, consider the sector with center at the origin, central angle θ, radius r, and the corresponding arc. Let R be the radial segment from the origin to the midpoint of the arc. Note that $\overline{x}_\gamma$ and $\overline{x}_\alpha$ lie on R (see FIGURE 4). In fact, using techniques from Calc. III, it can be shown that $\overline{x}_\alpha = \frac{2}{3}\overline{x}_\gamma$, where $\overline{x}_\gamma = kr$, for some quantity k independent of r. Hence, if S is the solid obtained by revolving the sector about the x-axis, and if ∂S denotes the surface generated by the arc only, the Theorems of Pappus yield

$$V = 2\pi\overline{x}_\alpha(\text{Area of sector}) = \frac{2}{3}\pi k\,r^3\theta$$

and

$$A = 2\pi\overline{x}_\gamma(\text{Length of arc}) = 2\pi k\,r^2\theta = \frac{dV}{dr}.$$

$\square$

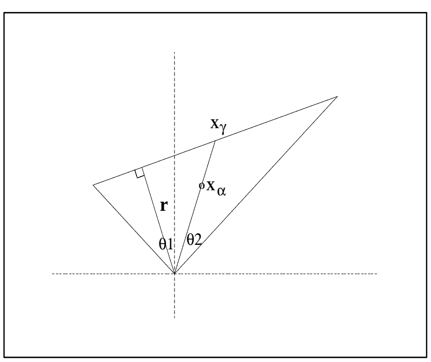

FIGURE 3: Triangular region.

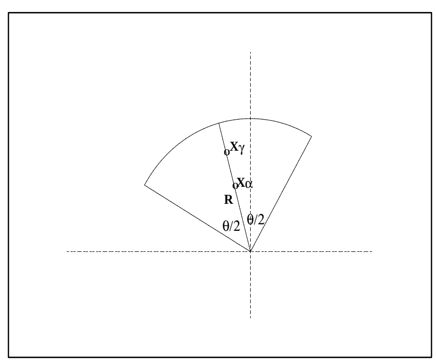

FIGURE 4: Sectorial region.

This theorem establishes that a right circular cone and a right circular cylinder with an inscribed sphere of radius r touching each side of the surface satisfy eq. (2).

Exercise 2. Show that $\frac{dV}{dr} = A$ for a right circular cone with an inscribed sphere of radius r and angle θ between the base and the side of the cone.

For a circle or a regular polygon, the region's centroid and the curve's centroid are the same. Thus we get the following corollary.

Corollary 8. *Let $R \in \mathbb{R}^2$ be a disc or a regular polygonal region with an inscribed circle of radius r and $S \in \mathbb{R}^3$ be the solid formed by revolving R about an axis that does not intersect R. Then*

$$\frac{d\,V}{d\,r} = A.$$

This gives us the interesting result that a torus satisfies eq. (2). Recall that a torus can be generated by rotating a circle centered at the point $(a, 0)$ and of radius r about the y-axis, where $r < a$. By Pappus' Theorems, $V = (2\pi a)(\pi r^2)$ and $A = (2\pi a)(2\pi r)$.

While a right circular cylinder can be formed by the approach in Theorem 7, it also can be created by lifting a circle into $\mathbb{R}^3$. This suggests the following theorem which can be proved by using Theorem 2, the Chain Rule, and the formulas $V = 2r \cdot (\text{Area of } R)$ and $A = 2 \cdot (\text{Area of } R) + 2r \cdot (\text{Perimeter of } R)$.

Theorem 9. *Let $R \in \mathbb{R}^2$ be a region which satisfies the conditions of Theorem 2 where r is the radius of the inscribed circle. Let $S \in \mathbb{R}^3$ be the solid formed by lifting R to a height of $2r$. Then*

$$\frac{d\,V}{d\,r} = A.$$

Intuitively, the $\mathbb{R}^3$ version of Lemma 1 replaces triangles with pyramids having polygonal bases. However, we can generalize this further.

Lemma 10. *Let S be a convex solid containing the origin O in $\mathbb{R}^3$. For every (smooth) component ∂S_i of the boundary ∂S assume that the position vector from O to any point q_i of ∂S_i has the form*

$$\vec{R}_i = r_i \vec{n}_i + \vec{p}_i$$

where $\vec{n}_i$ is the outward normal to ∂S_i at q_i and $\vec{p}_i \perp \vec{n}_i$. Then the generalized pyramid S_i with base ∂S_i and vertex O satisfies

$$V_{S_i} = \frac{1}{3} r_i A_{\partial S_i}$$

and

$$\frac{d\,V_{S_i}}{dr_i} = A_{\partial S_i}.$$

Proof. Apply the divergence theorem to the vector field $\vec{R}_i$ over S_i to get

$$3\,V_{S_i} = \int\int_{\partial S_i} \vec{R}_i \cdot \vec{n}_i \, d(\partial S)$$

or

$$V_{S_i} = \frac{1}{3} \int\int_{\partial S_i} r_i \, d(\partial S) = r_i A_{\partial S_i}.$$

Because r_i is a characteristic dimension of S_i, we have

$$A_{\partial S_i} = k_i r_i^2$$

and so

$$\frac{dV_{S_i}}{dr_i} = \frac{d}{dr_i}\left(\frac{1}{3}k_i r_i^3\right) = k_i r_i^2 = A_{\partial S_i}.$$

$\square$

Surfaces that satisfy $\vec{R}_i = r_i\vec{n}_i + \vec{p}_i$ with $\vec{p}_i \perp \vec{n}_i$ include right pyramids with polygonal bases, right circular cones, right circular cylinders, spheres, and planes. In some cases the location of O is restricted. For example, for a right circular cylinder or cone, O must be on the axis of the solid and in the interior, and for a sphere, O must be the center of the sphere.

By decomposing the solid S in $\mathbb{R}^3$ into finitely many of the regions described in Lemma 10, we get the analogue of Theorem 3.

Theorem 11. *Let S be a solid in $\mathbb{R}^3$ that is starlike with respect to some point O in the interior of S and whose boundary satisfies the hypotheses of Lemma 10. Then S satisfies*

$$\sum_{i=1}^{n} \frac{\partial V}{\partial r_i} = A,$$

where r_i is the altitude of the pyramid from O to the base side formed by the smooth surface of the generalized right pyramid.

If S has an inscribed sphere, $r_i = r$ for all i, and $\frac{dV}{dr} = A$. For any convex polyhedron S with n faces and O in its interior, if r_i is the perpendicular distance from O to the plane containing face ∂S_i, then

$$\sum_{i=1}^{n} \frac{\partial V}{\partial r_i} = A.$$

Also, Theorem 4 can be extended to $\mathbb{R}^3$ by a simple modification of its proof and noting that Tong's result in $\mathbb{R}^3$ is $r = 3V/A$.

Exercise 3. Show that $\sum_{i=1}^{3} \frac{\partial V}{\partial r_i} = A$ for a right circular cylinder of radius r_1 and height $r_2 + r_3$.

Example 1. Suppose n right circular cylinders, all having radius r, have their axes coplanar and concurrent. The possible solids, call them planar multi-cylinders, formed by the intersection of these cylinders and/or half-cylinders (half-cylinders have a D-shaped cross section with the straight side of the D vertical) include the convex shapes which can be made with the toy ODD BALLS®. For all such solids, W, there is an inscribed sphere of radius r and center O at the intersection of the axes, and we have

$$\frac{dV}{dr} = A.$$

If $n = 2$ and the axes intersect at right angles, the solid is the bicylinder studied by Hall's class (see also [2]), and the formulas are

$$V = \frac{16}{3}r^3 \qquad \text{and} \qquad A = 16\,r^2.$$

Formulas for other cases are left to the reader, but remember, the surface area is easily found once you have the volume.

Remark. The Archimedean property of the bicylinder also holds for any planar multi-cylinder and its circumscribed prism. Because both the multi-cylinder W and its circumscribed prism P have the same inscribed sphere of radius r, the theorem gives us that

$$V_W = \frac{1}{3} r\, A_W \qquad \text{and} \qquad V_P = \frac{1}{3} r\, A_P$$

from which it is clear that

$$\frac{V_W}{V_P} = \frac{A_W}{A_P}.$$

To show that these ratios are $\frac{2}{3}$, we will use a modification of the method in [2] for finding the surface areas of the pieces of the cylinders. Consider a wedge cut from a cylinder of radius r by a plane perpendicular to the axis and another plane intersecting the first plane on the axis at an angle α. In [2], $\alpha = \frac{\pi}{4}$. Then the surface cut off from the cylinder by these two planes has area

$$\int_0^{\pi r} r\, \tan\alpha \sin\left(\frac{u}{r}\right) du = 2r^2 \tan\alpha.$$

The faces of the circumscribing prism contained in the two intersecting planes will be in the interior of the multi-cylinder, so their areas are not required. The areas of the other three faces of the prism circumscribing the wedge are

$$A_P = (2r)(r\, \tan\alpha) + 2\left(\frac{1}{2}\right) r\, (r\, \tan\alpha) = 3r^2 \tan\alpha.$$

Thus, the ratio of the surface areas is $\frac{2}{3}$. The ratio is $\frac{2}{3}$ for any multi-cylinder because any multi-cylinder can be decomposed into a finite number of these wedges.

Exercise 4. Show that $\frac{dV}{dr} = A$ for the orthogonal tricylinder formed by intersecting three cylinders with radius r and concurrent mutually perpendicular axes.

Theorem 12. *Let P be a n-faced polyhedron that is starlike with respect to a point T in the interior of P such that P can be "triangulated" into n pyramids of equal volume with bases the faces of the polyhedron and opposite vertex T. If r_i is the distance from T to the ith face of P and r is the harmonic mean of $r_1, \ldots, r_n$, then*

$$\frac{d\,V}{d\,r} = A.$$

Example 2. Let S be a cylinder of radius r_1 and height $2r_2$. Let T be the point on the axis of S that is equidistant from the bases. This forms two cones of equal volume $\frac{1}{3}\pi r_1^2 r_2$ inside the cylinder by connecting T with each base. Slicing the cylinder with two mutually orthogonal planes whose intersection is the axis of S, the remaining interior of S is "triangulated" into four wedges, also of volume $\frac{1}{3}\pi r_1^2 r_2$. Then by Theorem 12, $dV/dr = A$, where $r = (3r_1 r_2)/(r_1 + 2r_2)$.

4 Areas For Further Investigation

Just as all the results in $\mathbb{R}^2$ carry over to $\mathbb{R}^3$, there are $\mathbb{R}^n$ analogues of the $\mathbb{R}^3$ results (see [1],[4]). However, Theorems 7 and 9 for $\mathbb{R}^3$ have no counterparts in $\mathbb{R}^2$, and it seems reasonable that there should be new theorems in $\mathbb{R}^n$ that have no analogues in $\mathbb{R}^{n-1}$. This is an area that could be further investigated.

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