Math 3325 **Extra Credit Homework (70 points)** **Spring 2007**

Problems 8–13 are due at the beginning of class on May 2, 2007.

It is quite obvious that statements and their connectives on the one hand, and sets and set connectives on the other hand behave somewhat analogously. The English mathematician George Boole *(1815–1864) made this idea precise by describing what he called "algebra of logic". Today we use the name "Boolean Algebra" in his honor instead:*

A Boolean Algebra is a set $\mathcal{B}$ together with two "connectives" $\sqcap$ and $\sqcup$ satisfying the following properties:

1. Closure Laws:
   (a) If A and B are two elements in $\mathcal{B}$, then $A \sqcap B$ is also an element in $\mathcal{B}$.
   (b) If A and B are two elements in $\mathcal{B}$, then $A \sqcup B$ is also an element in $\mathcal{B}$.
2. Commutative Laws:
   (a) $A \sqcap B = B \sqcap A$ for all elements A and B in $\mathcal{B}$.
   (b) $A \sqcup B = B \sqcup A$ for all elements A and B in $\mathcal{B}$.
3. Associative Laws:
   (a) $(A \sqcap B) \sqcap C = A \sqcap (B \sqcap C)$ for all elements A, B and C in $\mathcal{B}$.
   (b) $(A \sqcup B) \sqcup C = A \sqcup (B \sqcup C)$ for all elements A, B and C in $\mathcal{B}$.
4. Absorption Laws:
   (a) $A \sqcap (A \sqcup B) = A$ for all elements A and B in $\mathcal{B}$.
   (b) $A \sqcup (A \sqcap B) = A$ for all elements A and B in $\mathcal{B}$.
5. Distributive Laws:
   (a) $A \sqcap (B \sqcup C) = (A \sqcap B) \sqcup (A \sqcap C)$ for all elements A, B and C in $\mathcal{B}$.
   (b) $A \sqcup (B \sqcap C) = (A \sqcup B) \sqcap (A \sqcup C)$ for all elements A, B and C in $\mathcal{B}$.
6. There are elements $N \in \mathcal{B}$ (called the *null element*) and $O \in \mathcal{B}$ (the *one element*) such that
   (a) $A \sqcap N = N$ and $A \sqcap O = A$ for all elements A in $\mathcal{B}$.
   (b) $A \sqcup O = O$ and $A \sqcup N = A$ for all elements A in $\mathcal{B}$.
7. For every element A in $\mathcal{B}$ there is an element B in $\mathcal{B}$ such that $A \sqcap B = N$ and $A \sqcup B = O$.

Let X be an arbitrary set. Then $\mathcal{P}(X)$ with the connectives $\cap$ (in the role of $\sqcap$) and $\cup$ (in the role of $\sqcup$) forms a Boolean Algebra. Highlights of the proof are the subject of the problem below:

Problem 1 (old) Let X be an arbitrary set.

1. Show that the Absorption Laws **4a** and **4b** hold for $\mathcal{P}(X)$.

2. Which elements in $\mathcal{P}(X)$ play the role of the null element, and the one element, respectively (see Law **6**)?

3. For a given element A in $\mathcal{P}(X)$, how does one choose the element B mentioned in Law **7**?

Similarly certain sets of statements with connectives $\wedge$ (in the role of $\sqcap$) and $\vee$ (in the role of $\sqcup$) naturally form Boolean Algebras.

What is meant by "certain" sets of statements? Our task at hand is to identify what sets of statements correspond to power sets.

Let us consider an example and start with one "generic" statement P. How many distinct propositional forms can we form involving this statement? A little bit of reflection will lead us on the following path: Every propositional form has a truth table, so the number of distinct propositional forms is limited by the number of distinct truth tables. Since a truth table involving the statement P has two rows, and since we have two choices for each row entry (T or F), there are at most 4 distinct truth tables, and therefore there are at most 4 distinct propositional forms. On the other hand it is easy to see that P, $\neg P$, $P \vee \neg P$ and $P \wedge \neg P$ are 4 distinct propositional forms contained in each Boolean Algebra containing P.

It is now boring to check that the following 4-element set indeed forms a Boolean Algebra:

$$\mathcal{S}_1 = \{P \wedge \neg P;\ P,\ \neg P;\ P \vee \neg P\}$$

$\mathcal{S}_1$ is called the "Boolean Algebra generated by the free statement P".

Problem 2 (old) Find the Boolean Algebra $\mathcal{S}_2$ generated by two free statements P and Q.

*For a natural number n, let $\mathcal{D}_n$ denote the set of the divisors of n. For example, $\mathcal{D}_{42} = \{1, 2, 3, 6, 7, 14, 21, 42\}$ and $\mathcal{D}_{12} = \{1, 2, 3, 4, 6, 12\}$. For $m, n \in \mathbb{N}$ let $m \sqcap n$ denote the greatest common divisor of n and m, and $m \sqcup n$ their least common multiple. For instance $6 \sqcap 4 = 2$ and $6 \sqcup 4 = 12$. It turns out that $\mathcal{D}_{42}$ with these two operations $\sqcap$ and $\sqcup$ forms a Boolean Algebra, while $\mathcal{D}_{12}$ does **not**.*

Problem 3 (old) Verify Boolean Algebra Laws **5**, **6** and **7** for $\mathcal{D}_{42}$.

Problem 4 (old)

1. Show that $\mathcal{D}_{12}$ does not form a Boolean Algebra.
2. Conjecture for which values of n the set $\mathcal{D}_n$ forms a Boolean Algebra.

Problem 5 (old) Consider the relation "$\preceq$" on a Boolean Algebra $\mathcal{B}$ defined by

$$A \preceq B \quad \Leftrightarrow \quad A \sqcup B = B$$

for $A, B \in \mathcal{B}$. Prove that this relation is reflexive, anti-symmetric, and transitive.

Problem 6 (old) Consider the Boolean Algebra $\mathcal{S}_1$. Draw a Hasse diagram for $\mathcal{S}_1$ endowed with the partial order $\preceq$.

Problem 7 (old) Let $\mathcal{B}$ be a Boolean Algebra with null-element N, partially ordered by $\preceq$. We say that $A \in \mathcal{B}$ is an ATOM of $\mathcal{B}$ if N is the immediate predecessor of A.

1. Find all atoms of $\mathcal{P}(\{1,2,3,4\})$.
2. Find all atoms of $\mathcal{D}_{42}$.
3. Assume additionally that $\mathcal{B}$ has finitely many elements. Show that for every $B \in \mathcal{B}$ with $B \neq N$ there is an atom A such that $A \preceq B$.

Problem 8 (10 points) Find a Boolean Algebra with 8 elements that is a subset of $\mathcal{P}(\{1,2,3,4\})$, but **not** the power set of a three-element subset of $\{1,2,3,4\}$, then find its atoms and draw its Hasse diagram.

Problem 9 (10 points) Let A, B be two elements in a Boolean Algebra. Show the following:

1. $A \preceq B \Leftrightarrow A \sqcap B = A$.
2. If $A \sqcap B = N$ and $A \preceq B$, then $A = N$.

Problem 10 (10 points) Let A_1, A_2 be two atoms in a finite Boolean Algebra. Show the following:

1. The least upper bound of the set $\{A_1, A_2\}$ is the element $A_1 \sqcup A_2$.
2. If $A_1 \neq A_2$, then $A_1 \sqcap A_2 = N$.

Problem 11 (10 points) Given an element B in a finite Boolean Algebra $\mathcal{B}$, we let

$$\alpha(B) = \{A \in \mathcal{B} \mid A \preceq B \text{ and } A \text{ is an atom of } \mathcal{B}\}.$$

(By Problem 7.3 we know that $\alpha(B) \neq \emptyset$ whenever $B \neq N$. Also note that $\alpha(N) = \emptyset$.) Let $A_1 \neq A_2$ be two atoms in $\mathcal{B}$. Show that $\alpha(A_1 \sqcup A_2) = \{A_1, A_2\}$.

In the sequel, you may assume that results corresponding to those proved for two atoms in Problems 10 and 11 also hold for finitely many atoms.

Problem 12 (10 points) Let $B \neq N$ be an element in a finite Boolean Algebra $\mathcal{B}$, and suppose $\alpha(B) = \{A_1, A_2, A_3, \ldots, A_k\}$ for some $k \in \mathbb{N}$ and some atoms $A_1, A_2, A_3, \ldots, A_k$ of $\mathcal{B}$. Show:

$$B = A_1 \sqcup A_2 \sqcup A_3 \sqcup \ldots \sqcup A_k.$$

Hint: Expect to use Boolean Algebra Law **7** along the way.

The last problem is the finite version of a general representation theorem for Boolean Algebras, proved in 1934 by the American mathematician MARSHALL H. STONE *(1903–1989):*

Problem 13 (20 points) Let $\mathcal{B}$ be a finite Boolean Algebra with k atoms for some $k \in \mathbb{N}$, and let $\mathcal{A}$ denote the power set of the set of all atoms of $\mathcal{B}$.

1. Show that the function $\alpha : \mathcal{B} \to \mathcal{A}$, defined above, is a bijection.
2. $\mathcal{B}$ has 2^k elements.
3. Show that the identities $\alpha(B \sqcup B') = \alpha(B) \cup \alpha(B')$ and $\alpha(B \sqcap B') = \alpha(B) \cap \alpha(B')$ hold for all $B, B' \in \mathcal{B}$.