The assignment is due at the beginning of class on April 9, 2007.

Problem 1 (10 points) On the set of natural numbers $\mathbb{N}$ consider the partial order

$$n \mid m \Leftrightarrow n \text{ is a divisor of } m.$$

1. Draw a *Hasse diagram* for the set $A = \{1, 2, 3, 4, 5, \ldots, 12, 13, 14, 15\}$ endowed with this partial order.
2. Find three upper bounds for A.
3. Find the largest element of A, or show that it does not exist.
4. Find the maximal elements of A, or show that none exist.

Problem 2 (10 points) Consider $\mathbb{R}^2$ with the lexicographical order $\leq_\ell$:

$$(a, b) \leq_\ell (c, d) \quad \Leftrightarrow \quad (a < c) \vee ((a = c) \wedge (b \leq d)).$$

1. Show that $\leq_\ell$ is a linear order on $\mathbb{R}^2$.
2. Find a subset of $(\mathbb{R}^2, \leq_\ell)$ that is bounded from above, but fails to have a least upper bound.

Problem 3 (10 points) Consider the relation "$\preceq$" on a Boolean Algebra $\mathcal{B}$ defined by

$$A \preceq B \quad \Leftrightarrow \quad A \sqcup B = B$$

for $A, B \in \mathcal{B}$. You already proved in Homework 7 that this relation is reflexive and transitive. Prove that $\preceq$ is also anti-symmetric.

Problem 4 (10 points) Consider the Boolean Algebra $\mathcal{S}_1$, defined in Homework 4. Draw a *Hasse diagram* for $\mathcal{S}_1$ endowed with the partial order $\preceq$.

Problem 5 (10 points) Let $\mathcal{B}$ be a Boolean Algebra with null-element N, partially ordered by $\preceq$. We say that $A \in \mathcal{B}$ is an ATOM of $\mathcal{B}$ if N is the immediate predecessor of A.

1. Find all atoms of $\mathcal{P}(\{1, 2, 3, 4\})$.
2. Find all atoms of $\mathcal{D}_{42}$, defined in Homework 5.
3. Assume additionally that $\mathcal{B}$ has finitely many elements. Show that for every $B \in \mathcal{B}$ with $B \neq N$ there is an atom A such that $A \preceq B$.