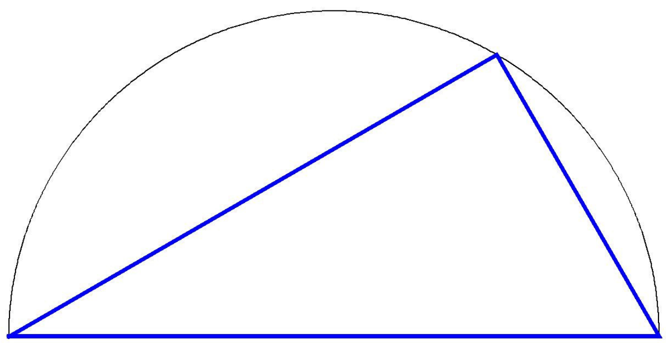
B A Short History of Proof

While mathematical knowledge is as old as written history, the Greek philosopher, mathematician and engineer THALES OF MILETUS (∼624 B.C.–∼547 B.C.) is usually credited with inventing the method of proof. None of his writings survived, but the following result is known as the "Theorem of Thales": *Every triangle with its three vertices on a common circle such two of the points are diametrically opposed is a right triangle.*



Theorem of Thales

By the time EUCLID OF ALEXANDRIA (∼325 B.C–∼265 B.C.) wrote his compendium of then current mathematical knowledge, called *The Elements*[10], a theory of proof had been established: A mathematical theory starts with "primitive definitions" and "axioms". Axioms are statements which are "self-evident" (statements that need no proof/are not proved). All other results of the mathematical theory are then deduced from these axioms with the help of logical reasoning.

For example, Book I of *The Elements* starts with the following two primitive definitions: A *point* is that which has no part; a *line* is breadthless width. Euclid's Axiom 5 is known as the "Parallel Axiom": *Given a line ℓ and a point P not on line ℓ, there is a unique line that is parallel to*

[10] An annotated version by David Joyce can be found at `http://aleph0.clarku.edu/~djoyce/java/elements/toc.html`

line ℓ and that passes through the point P[11].

An axiomatic system needs to be *consistent* (the axioms do not lead to contradictory results). It is also desirable that an axiomatic system is *complete* (all statements within the theory can, at least in principle, be shown to be true or false) and *minimal* (it is not possible to deduce one of the axioms from the other axioms).

The axiomatic method described by Euclid still works pretty well for "working mathematicians" today; it will basically be the method of proof used in this course.

Historically, numerous mathematicians came to question whether Euclid's axiomatic system for geometry is minimal. The Parallel Axiom was the last axiom Euclid wrote down, so perhaps he himself thought that this axiom could maybe be deduced from his other axioms. No such proof could be found in nearly 2,000 years. Finally, Nikolai Lobachevsky (1792–1856), János Bolyai (1802–1860) and Johann Carl Friedrich Gauss (1777–1855) independently discovered models for Non-euclidean Geometry, in which all axioms of Euclidean Geometry hold except for the Parallel Axiom (whose negation is assumed to be true), thereby establishing the independence of Axiom 5 from the other axioms Euclid had postulated.

Euclid's idea seems to have been to create different axiomatic systems for different branches of mathematics. The obvious question arises whether all areas of mathematics can be reduced to a hierarchical system of axioms.

Gottfried Wilhelm von Leibniz (1646–1746), the co-discoverer of Calculus, dreamed of such a universal symbolic language in which all human thought could be expressed[12]. More concrete steps were undertaken by George Boole (1815–1864) to create a symbolic language for logic. Finally in 1879, Gottlob Frege (1848–1925) published his

[11] Euclid actually phrased this axiom differently.

[12] It is interesting to note in this context that Leibniz was the first mathematician to consider the binary number system—after all it is the "minimal" system in which information can be stored.

book *Begriffsschrift*. The complete title in English translation: *Conceptual Notation, a formal language modelled on that of arithmetic, for pure thought.* In *Begriffsschrift*, Frege developed a complete formal system for logical reasoning, with definitions and notations still in common use today (see Section 1.1). He then outlined a plan to build the whole of mathematics on this foundation.

The next steps in this program were pursued by Georg Cantor (1845–1918) and Richard Dedekind (1831–1916). Cantor created set theory (see Section 1.2) and studied the many surprising phenomena encountered when considering infinite collection of objects (see Section 4); Dedekind gave an axiomatic description of elementary arithmetic and constructed the real numbers from basic axioms for the counting numbers (see Section 2.1).

Frege's program collapsed on June 16, 1902, when he received a letter from Bertrand Russell (1872–1970) stating that Russell had found a contradiction in Frege's axiomatic system.

Russell's paradox goes as follows: Starting with Frege's axioms, Russell constructs the "set" C of all sets that do not contain themselves as elements. Now suppose C is an element of itself; this contradicts the definition of C of consisting of all sets that do **not** contain themselves. If, on the other hand, C is not an element of itself, then it consequently fits the membership description of elements in C, and is thus contained in C.

Frege was unable to "fix" this fatal flaw; Russell together with Alan N. Whitehead (1861–1947) worked on a solution in their voluminous *Principia Mathematica*; they themselves were unsatisfied with the "elegance" of their work-arounds; a plan for a fourth and final volume of *Principia Mathematica* was abandoned. This brought an end to "Logicism" as Frege's program has become known as.