Theorem. $\sqrt{2}$ is irrational.

Proof. Suppose to the contrary that $\sqrt{2}$ is rational. Then we can find natural numbers p and q such that $p^2 = 2q^2$.

In base 3, the last non-zero digit of every natural number is either 1 or 2, thus p can be written in the form (1) $p = 3^k(3n+1)$ for some $k \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N} \cup \{0\}$, or (2) $p = 3^k(3n+2)$ for some $k \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N} \cup \{0\}$.

If (1) holds, then $p^2 = 3^{2k}(9n^2 + 6n + 1) = 3^{2k+1}(3n^2 + 2n) + 1 \cdot 3^{2k}$; if (2) holds, it follows that $p^2 = 3^{2k}(9n^2 + 12n + 4) = 3^{2k+1}(3n^2 + 4n + 1) + 1 \cdot 3^{2k}$.

In other words, written in base 3, the last non-zero digit of p^2 equals 1. Similarly, the last non-zero digit of q^2 in base 3 equals 1, and so the last non-zero digit of $2q^2$ in base 3 equals 2.

Consequently our assumption $p^2 = 2q^2$ must be false.

Exercise 1. Show that in base 10, the last non-zero digit of the square of a natural number equals 1,4,5,6, or 9.

Exercise 2. Can you modify the proof above to work in base 2?