Math 3341 Introduction to Analysis

Helmut Knaust

Department of Mathematical Sciences
The University of Texas at El Paso
El Paso TX 79968-0514

hknaust@utep.edu

Summer II 2007

Last edit: July 14, 2007

Task 1.1

Let $P(n)$ be a predicate with domain $\mathbb{N}$. If

1. *$P(1)$ is true, and*
2. *Whenever $P(n)$ is true, then $P(n+1)$ is true,*

then $P(n)$ is true for all $n \in \mathbb{N}$.

Task 1.2

Show that the square root of 2 is irrational. ($\sqrt{2}$ is the positive real number whose square is 2.)

Exercise 1.3

Write down the axioms **G1–G5** *explicitly for the multiplicative group* $(\mathbb{Q} \setminus \{0\}, \cdot)$.

Exercise 1.4

Show that a set can have at most one maximum.

Exercise 1.4

Show that a set can have at most one maximum.

Exercise 1.5

Characterize all subsets A of the set of real numbers with the property that $\min A = \max A$.

Exercise 1.4

Show that a set can have at most one maximum.

Exercise 1.5

Characterize all subsets A of the set of real numbers with the property that $\min A = \max A$.

Task 1.6

Show that finite non-empty sets of real numbers always have a minimum.

Exercise 1.7

For all $a, b \in \mathbb{R}$*:*

$$|a+b| \leq |a| + |b|$$

Exercise 1.7

For all $a, b \in \mathbb{R}$*:*

$$|a+b| \leq |a| + |b|$$

Exercise 1.8

For all $a, b \in \mathbb{R}$*:*

$$|a-b| \geq \Big||a| - |b|\Big|$$

Exercise 1.9

Show that for every positive real number r, there is a natural number n, such that $0 < \frac{1}{n} < r$.

Exercise 1.9

Show that for every positive real number r, there is a natural number n, such that $0 < \frac{1}{n} < r$.

Task 1.10

The set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$.

Exercise 1.9

Show that for every positive real number r, there is a natural number n, such that $0 < \frac{1}{n} < r$.

Task 1.10

The set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$.

Task 1.11

The set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

Exercise 2.1

Let $(a_n)_{n\in\mathbb{N}}$ denote the sequence of prime numbers in their natural order. What is a_5?

Exercise 2.1

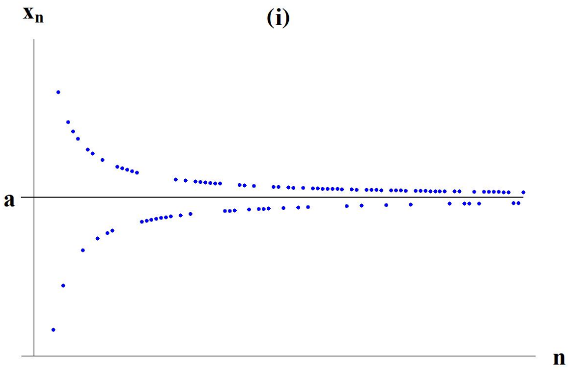
Let $(a_n)_{n\in\mathbb{N}}$ *denote the sequence of prime numbers in their natural order. What is* a_5*?*

Exercise 2.2

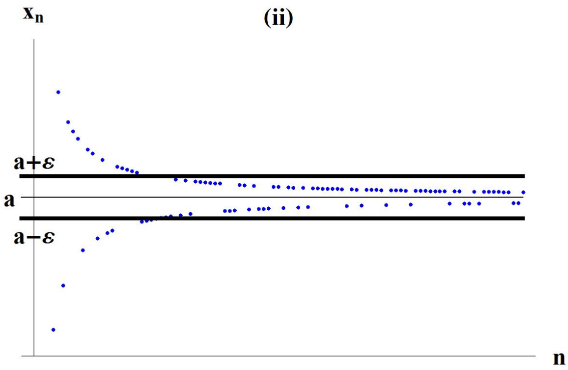
Write the sequence $0, 1, 0, 2, 0, 3, 0, 4, \ldots$ *as a function* $\varphi : \mathbb{N} \to \mathbb{R}$.

Exercise 2.3

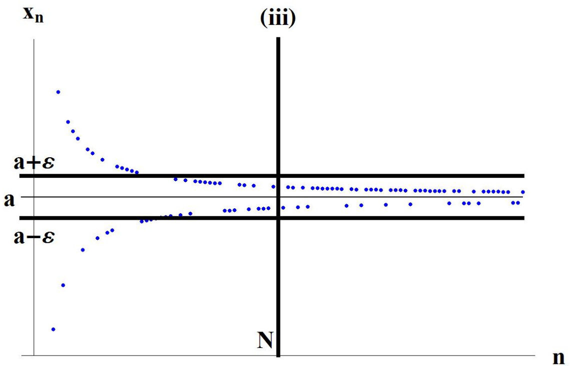
Spend some quality time studying the figure on the next slide. Explain how the pictures and the parts in the definition correspond to each other. Also reflect on how the "rigorous" definition above relates to your prior understanding of what it means for a sequence to converge.



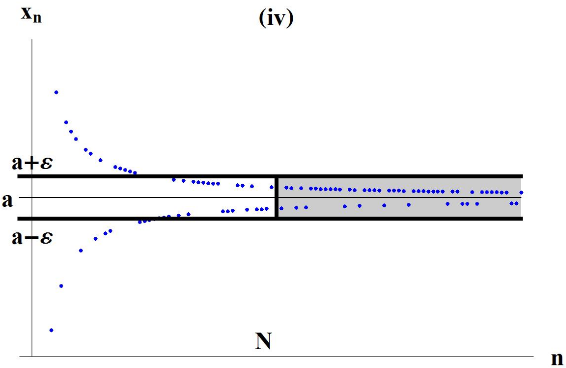
(i) A sequence (x_n) converges to the limit a if . . .



(i) A sequence (x_n) converges to the limit a if . . . (ii) . . . for all $\varepsilon > 0$ . . .



(i) A sequence (x_n) converges to the limit a if ... (ii) ... for all $\varepsilon > 0$ ... (iii)... there is an $N \in \mathbb{N}$, such that ...



(i) A sequence (x_n) converges to the limit a if ... (ii) ... for all $\varepsilon > 0$ ... (iii)... there is an $N \in \mathbb{N}$, such that ... (iv) ... $|x_n - a| < \varepsilon$ for all $n \geq N$.

Exercise 2.4

1. *Write down formally (using ε-N language) what it means that a given sequence $(a_n)_{n\in\mathbb{N}}$ does not converge to the real number a.*
2. *Similarly, write down what it means for a sequence to diverge.*

Exercise 2.5

Show that the sequence $a_n = \dfrac{(-1)^n}{\sqrt{n}}$ *converges to* 0.

Exercise 2.5

Show that the sequence $a_n = \dfrac{(-1)^n}{\sqrt{n}}$ *converges to* 0.

Exercise 2.6

Show that the sequence $a_n = 1 - \dfrac{1}{n^2+1}$ *converges to* 1.

Task 2.7

Show: If a sequence converges to two real numbers a and b, then $a = b$.

Exercise 2.8

Give an example of a bounded sequence which does not converge.

Exercise 2.8

Give an example of a bounded sequence which does not converge.

Task 2.9

Every convergent sequence is bounded.

Task 2.10

If the sequence (a_n) converges to a, and the sequence (b_n) converges to b, then the sequence $(a_n + b_n)$ is also convergent and its limit is $a + b$.

Task 2.10

If the sequence (a_n) *converges to* a*, and the sequence* (b_n) *converges to* b*, then the sequence* $(a_n + b_n)$ *is also convergent and its limit is* $a + b$*.*

Task 2.11

If the sequence (a_n) *converges to* a*, and the sequence* (b_n) *converges to* b*, then the sequence* $(a_n \cdot b_n)$ *is also convergent and its limit is* $a \cdot b$*.*

Task 2.12

Let (a_n) be a sequence converging to $a \neq 0$. Then there are a $\delta > 0$ and an $M \in \mathbb{N}$ such that $|a_m| > \delta$ for all $m \geq M$.

Task 2.12

Let (a_n) be a sequence converging to $a \neq 0$. Then there are a $\delta > 0$ and an $M \in \mathbb{N}$ such that $|a_m| > \delta$ for all $m \geq M$.

Task 2.13

Let the sequence (b_n) with $b_n \neq 0$ for all $n \in \mathbb{N}$ converge to $b \neq 0$. Then the sequence $\left(\dfrac{1}{b_n}\right)$ is also convergent and its limit is $\dfrac{1}{b}$.

Task 2.14

Let (a_n) *be a sequence converging to* a*. If* $a_n \geq 0$ *for all* $n \in \mathbb{N}$*, then* $a \geq 0$*.*

Task 2.15

Let $a_1 = 1$ and $a_{n+1} = \sqrt{2a_n + 1}$ for all $n \in \mathbb{N}$. Show that the sequence (a_n) converges.

Exercise 2.16

Find the supremum of each of the following sets:

1. *The closed interval* $[-2, 3]$

Exercise 2.16

Find the supremum of each of the following sets:

1. *The closed interval* $[-2, 3]$
2. *The open interval (0,2)*

Exercise 2.16

Find the supremum of each of the following sets:

1. *The closed interval* $[-2, 3]$
2. *The open interval (0,2)*
3. *The set* $\{x \in \mathbb{Z} \mid x^2 < 5\}$

Exercise 2.16

Find the supremum of each of the following sets:

1. *The closed interval* $[-2, 3]$
2. *The open interval (0,2)*
3. *The set* $\{x \in \mathbb{Z} \mid x^2 < 5\}$
4. *The set* $\{x \in \mathbb{Q} \mid x^2 < 3\}$.

Exercise 2.17

Let (a_n) be an increasing bounded sequence. By the Completeness Axiom the sequence converges to some real number a. Show that its range $\{a_n \mid n \in \mathbb{N}\}$ has a supremum, and that the supremum equals a.

Exercise 2.17

Let (a_n) *be an increasing bounded sequence. By the Completeness Axiom the sequence converges to some real number* a*. Show that its range* $\{a_n \mid n \in \mathbb{N}\}$ *has a supremum, and that the supremum equals* a*.*

Task 2.18

The Completeness Axiom is equivalent to the following: Every non-empty set of real numbers which is bounded from above has a supremum.

Exercise 2.19

Let $(a_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$*. Which of the following sequences are subsequences of* $(a_n)_{n\in\mathbb{N}}$*?*

1. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \ldots$

For the subsequence examples, also find the function $\psi : \mathbb{N} \to \mathbb{N}$.

Exercise 2.19

Let $(a_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$. *Which of the following sequences are subsequences of* $(a_n)_{n\in\mathbb{N}}$*?*

1. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \ldots$
2. $\frac{1}{2}, 1, \frac{1}{4}, \frac{1}{3}, \frac{1}{6}, \frac{1}{5} \ldots$

For the subsequence examples, also find the function $\psi : \mathbb{N} \to \mathbb{N}$.

Exercise 2.19

Let $(a_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$. *Which of the following sequences are subsequences of* $(a_n)_{n\in\mathbb{N}}$*?*

1. $1, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \dfrac{1}{5} \dots$
2. $\dfrac{1}{2}, 1, \dfrac{1}{4}, \dfrac{1}{3}, \dfrac{1}{6}, \dfrac{1}{5} \dots$
3. $1, \dfrac{1}{3}, \dfrac{1}{6}, \dfrac{1}{10}, \dfrac{1}{15} \dots$

For the subsequence examples, also find the function $\psi : \mathbb{N} \to \mathbb{N}$.

Exercise 2.19

Let $(a_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$. *Which of the following sequences are subsequences of* $(a_n)_{n\in\mathbb{N}}$*?*

1. $1, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \dfrac{1}{5} \dots$
2. $\dfrac{1}{2}, 1, \dfrac{1}{4}, \dfrac{1}{3}, \dfrac{1}{6}, \dfrac{1}{5} \dots$
3. $1, \dfrac{1}{3}, \dfrac{1}{6}, \dfrac{1}{10}, \dfrac{1}{15} \dots$
4. $1, 1, \dfrac{1}{3}, \dfrac{1}{3}, \dfrac{1}{5}, \dfrac{1}{5} \dots$

For the subsequence examples, also find the function $\psi : \mathbb{N} \to \mathbb{N}$.

Task 2.20

If a sequence converges, then all of its subsequences converge to the same limit.

Task 2.20

If a sequence converges, then all of its subsequences converge to the same limit.

Task 2.21

Show that every sequence of real numbers has an increasing subsequence or it has a decreasing subsequence.

Task 2.20

If a sequence converges, then all of its subsequences converge to the same limit.

Task 2.21

Show that every sequence of real numbers has an increasing subsequence or it has a decreasing subsequence.

Task 2.22 (Bolzano-Weierstrass)

Every bounded sequence of real numbers has a converging subsequence.

Task 2.23

*Suppose the sequence (a_n) does **not** converge to the real number L. Then there is an $\varepsilon > 0$ and a subsequence (a_{n_k}) of (a_n) such that*

$$|a_{n_k} - L| \geq \varepsilon \text{ for all } k \in \mathbb{N}.$$

Task 2.23

*Suppose the sequence (a_n) does **not** converge to the real number L. Then there is an $\varepsilon > 0$ and a subsequence (a_{n_k}) of (a_n) such that*

$$|a_{n_k} - L| \geq \varepsilon \text{ for all } k \in \mathbb{N}.$$

Task 2.24

*Let (a_n) be a **bounded** sequence. Suppose all of its **convergent** subsequences converge to the same limit a. Then (a_n) itself converges to a.*

Exercise 2.25

Every convergent sequence is a Cauchy sequence.

Exercise 2.25

Every convergent sequence is a Cauchy sequence.

Exercise 2.26

Every Cauchy sequence is bounded.

Task 2.27

If a Cauchy sequence has a converging subsequence with limit a, then the Cauchy sequence itself converges to a.

Task 2.27

If a Cauchy sequence has a converging subsequence with limit a, then the Cauchy sequence itself converges to a.

Task 2.28

Every Cauchy sequence is convergent.

Task 2.29

A sequence (a_n) converges to $L \in \mathbb{R}$ if and only if every neighborhood of L contains all but a finite number of the terms of the sequence (a_n).

Task 2.29

A sequence (a_n) converges to $L \in \mathbb{R}$ if and only if every neighborhood of L contains all but a finite number of the terms of the sequence (a_n).

Task 2.30

The real number x is an accumulation point of the set S if and only if every neighborhood of x contains an element of S different from x.

Exercise 2.31

Find all accumulation points of the following sets:

1. $\mathbb{Q}$

Exercise 2.31

Find all accumulation points of the following sets:

1. $\mathbb{Q}$
2. $\mathbb{N}$

Exercise 2.31

Find all accumulation points of the following sets:

1. $\mathbb{Q}$
2. $\mathbb{N}$
3. $[a, b)$

Exercise 2.31

Find all accumulation points of the following sets:

1. $\mathbb{Q}$
2. $\mathbb{N}$
3. $[a, b)$
4. $\left\{ \dfrac{1}{n} \mid n \in \mathbb{N} \right\}$.

Exercise 2.32

1. *Find a set of real numbers with exactly two accumulation points.*

Exercise 2.32

1. *Find a set of real numbers with exactly two accumulation points.*
2. *Find a set of real numbers whose accumulation points form a sequence* (a_n) *with* $a_n \neq a_m$ *for all* $n \neq m$*.*

Task 2.33

Show that x is an accumulation point of the set S if and only if there is a sequence (x_n) of elements in S with $x_n \neq x_m$ for all $n \neq m$ such that (x_n) converges to x.

Task 2.33

Show that x is an accumulation point of the set S if and only if there is a sequence (x_n) of elements in S with $x_n \neq x_m$ for all $n \neq m$ such that (x_n) converges to x.

Task 2.34

Every infinite bounded set of real numbers has at least one accumulation point.

Task 2.35

Let S be a non-empty set of real numbers which is bounded from above. Show: If $\sup S \notin S$, then $\sup S$ is an accumulation point of S.

Definition

Let $D \subseteq \mathbb{R}$, let $f : D \to \mathbb{R}$ be a function.

We say that the LIMIT of $f(x)$ at x_0 is equal to $L \in \mathbb{R}$, if for all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon$$

whenever $x \in D$ and $|x - x_0| < \delta$.
In this case we write $\lim_{x \to x_0} f(x) = L$.

Definition

Let $D \subseteq \mathbb{R}$, let $f : D \to \mathbb{R}$ be a function and let x_0 be an accumulation point of D.
We say that the LIMIT of $f(x)$ at x_0 is equal to $L \in \mathbb{R}$, if for all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon$$

whenever $x \in D$ and $\quad |x - x_0| < \delta$.
In this case we write $\lim\limits_{x \to x_0} f(x) = L$.

Definition

Let $D \subseteq \mathbb{R}$, let $f : D \to \mathbb{R}$ be a function and let x_0 be an accumulation point of D.
We say that the LIMIT of $f(x)$ at x_0 is equal to $L \in \mathbb{R}$, if for all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon$$

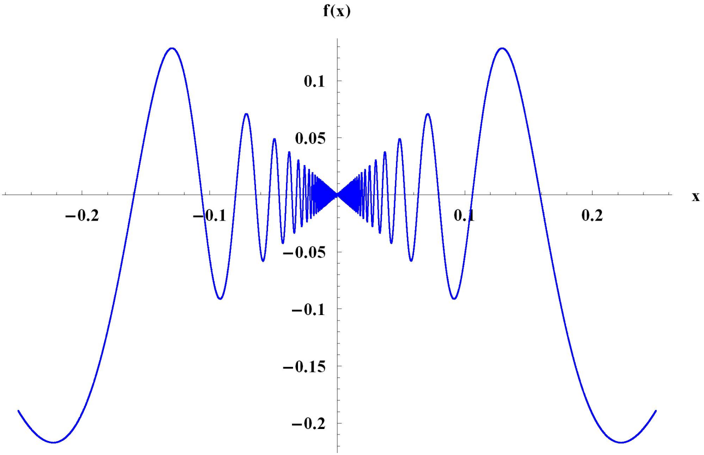
whenever $x \in D$ and $0 < |x - x_0| < \delta$.
In this case we write $\lim\limits_{x \to x_0} f(x) = L$.

Exercise 3.1

Let $f : \mathbb{R} \to \mathbb{R}$ *be defined by*

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0, \; x \in \mathbb{R} \\ 0, & \text{if } x = 0 \end{cases}$$

Does $f(x)$ *have a limit a* $x_0 = 0$*? If so, what is the limit?*


f(x)
x
−0.2
−0.1
0.1
0.2
0.1
0.05
−0.05
−0.1
−0.15
−0.2

Exercise 3.2

Let $D \subseteq \mathbb{R}$, let $f : D \to \mathbb{R}$ be a function and let x_0 be an accumulation point of D. Then the following are equivalent:

1. $\lim_{x \to x_0} f(x)$ *exists and is equal to L.*
2. *Let (x_n) be any sequence of elements in D that converges to x_0, and satisfies that $x_n \neq x_0$ for all $n \in \mathbb{N}$. Then the sequence $f(x_n)$ converges to L.*

Exercise 3.2

Let $D \subseteq \mathbb{R}$, let $f : D \to \mathbb{R}$ be a function and let x_0 be an accumulation point of D. Then the following are equivalent:

1. $\lim_{x \to x_0} f(x)$ *exists and is equal to L.*
2. *Let (x_n) be any sequence of elements in D that converges to x_0, and satisfies that $x_n \neq x_0$ for all $n \in \mathbb{N}$. Then the sequence $f(x_n)$ converges to L.*

Exercise 3.3

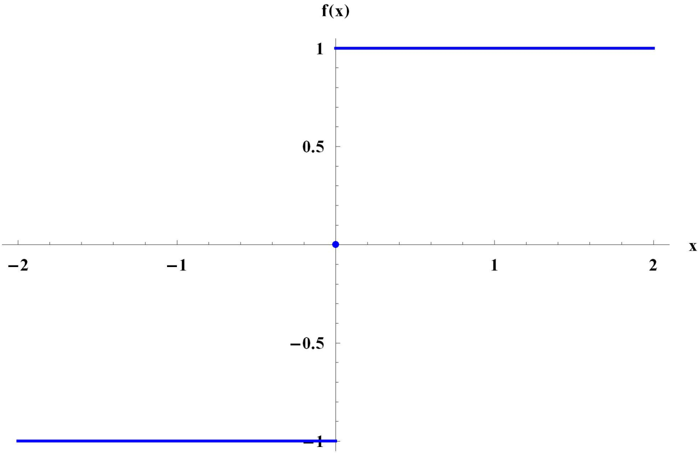
What strategy does Exercise 3.2 suggest to show non-existence of a limit at a point?

Exercise 3.4

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} |x|/x, & \text{if } x \neq 0,\ x \in \mathbb{R} \\ 0, & \text{if } x = 0 \end{cases}$$

Does $f(x)$ have a limit a $x_0 = 0$? If so, what is the limit?

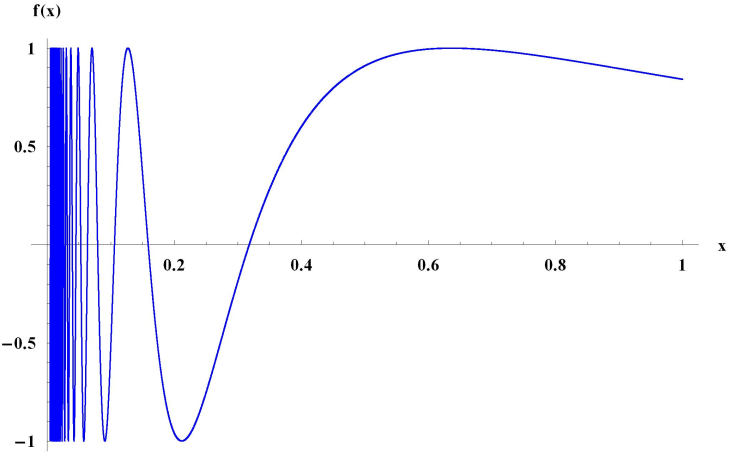

f(x)
1
0.5
−0.5
−1
x
−2
−1
1
2

Exercise 3.5

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0, \ x \in \mathbb{R} \\ 0, & \text{if } x = 0 \end{cases}$$

Does $f(x)$ have a limit a $x_0 = 0$? If so, what is the limit?


f(x)
1
0.5
−0.5
−1
0.2
0.4
0.6
0.8
1
x

Exercise 3.6

Let $f : (0,1] \to \mathbb{R}$ *be defined by*

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

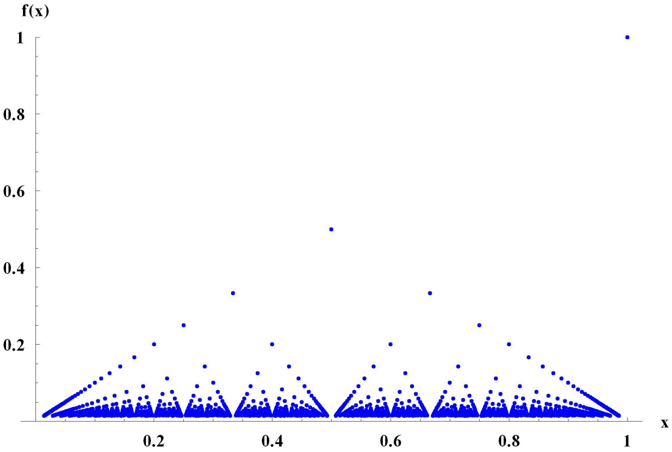
For which values of x_0 *does* $f(x)$ *have a limit a* x_0*? What is the limit?*

Task 3.7

Let $f : (0,1] \to \mathbb{R}$ *be defined by*

$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ with } p, q \text{ relatively prime} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

For which values of x_0 *does* $f(x)$ *have a limit a* x_0*? What is the limit?*


f(x)
1
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1
x

Exercise 3.8

Let $D \subseteq \mathbb{R}$, let $f : D \to \mathbb{R}$ be a function and let x_0 be an accumulation point of D.
If $f(x)$ has a limit at x_0, then there is a $\delta > 0$ and an $M > 0$ such that

$$|f(x)| \leq M \text{ for all } x \in (x_0 - \delta, x_0 + \delta) \cap D.$$

Definition

Let D be a set of real numbers and $x_0 \in D$. A function $f : D \to \mathbb{R}$ is said to be CONTINUOUS at x_0 if the following holds: For all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in D$ with

$$|x - x_0| < \delta,$$

we have that

$$|f(x) - f(x_0)| < \varepsilon.$$

Exercise 4.1

Let D be a set of real numbers and $x_0 \in D$ be an accumulation point of D. Then the function $f : D \to \mathbb{R}$ is continuous at x_0 if and only if $\lim_{x \to x_0} f(x) = f(x_0)$.

Exercise 4.1

Let D be a set of real numbers and $x_0 \in D$ be an accumulation point of D. Then the function $f : D \longrightarrow \mathbb{R}$ is continuous at x_0 if and only if $\lim_{x \longrightarrow x_0} f(x) = f(x_0)$.

Exercise 4.2

Let D be a set of real numbers and $x_0 \in D$. Assume also that x_0 is not an accumulation point of D. Then the function $f : D \longrightarrow \mathbb{R}$ is continuous at x_0.

Exercise 4.3

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} |x|, & \text{if } x \in \mathbb{Q} \\ x^2, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

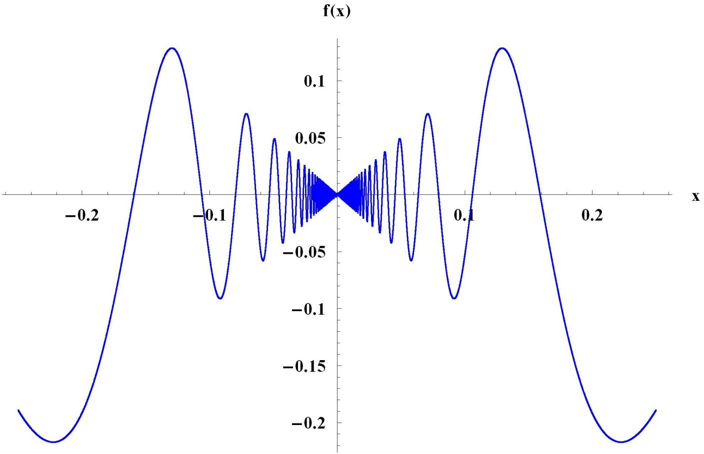
For which values of x_0 is $f(x)$ continuous?

Exercise 4.4

Let $f : \mathbb{R} \to \mathbb{R}$ *be defined by*

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0,\ x \in \mathbb{R} \\ 0, & \text{if } x = 0 \end{cases}$$

Is $f(x)$ *continuous at* $x_0 = 0$*?*

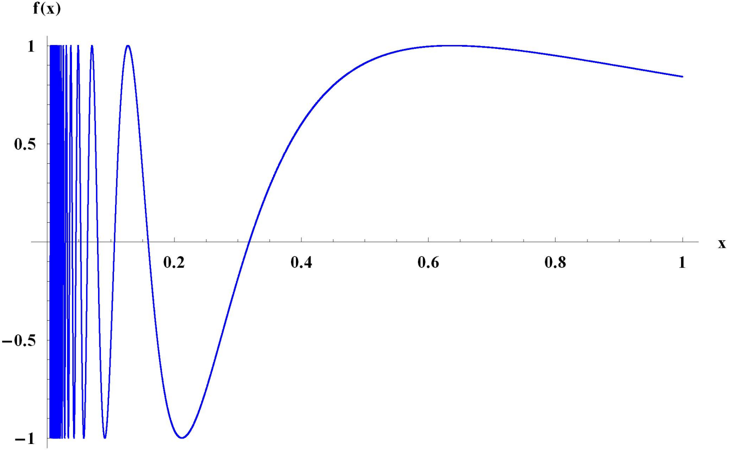

f(x)
0.1
0.05
−0.05
−0.1
−0.15
−0.2
x
−0.2
−0.1
0.1
0.2

Exercise 4.5

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & \textit{if } x \neq 0,\ x \in \mathbb{R} \\ 0, & \textit{if } x = 0 \end{cases}$$

Is $f(x)$ continuous at $x_0 = 0$?


f(x)
1
0.5
−0.5
−1
0.2
0.4
0.6
0.8
1
x

Exercise 4.6

Let $f : (0,1] \to \mathbb{R}$ *be defined by*

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

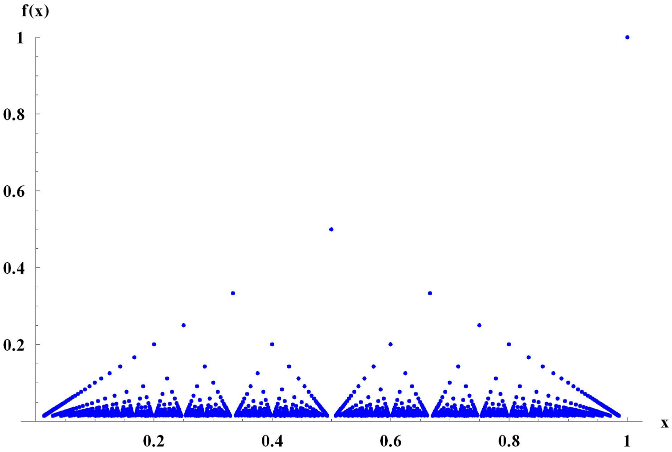
For which values of x_0 *is* $f(x)$ *continuous?*

Exercise 4.7

Let $f : (0, 1] \to \mathbb{R}$ *be defined by*

$$f(x) = \begin{cases} \frac{1}{q}, & \textit{if } x = \frac{p}{q} \textit{ with } p, q \textit{ relatively prime} \\ 0, & \textit{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

For which values of x_0 *is* $f(x)$ *continuous?*


f(x)
1
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1
x

Task 4.8

Let $D, E \subseteq \mathbb{R}$, and let $f : D \to \mathbb{R}$ be a function continuous at $x_0 \in D$. Assume $f(D) \subseteq E$. Suppose $g : E \to \mathbb{R}$ is a function continuous at $f(x_0)$. Then the composition $g \circ f : D \to \mathbb{R}$ is continuous at x_0.

Exercise 4.9

If $f : D \to \mathbb{R}$ is uniformly continuous on D, then f is continuous on D. What is the difference between continuity and uniform continuity?

Exercise 4.9

If $f : D \to \mathbb{R}$ is uniformly continuous on D, then f is continuous on D. What is the difference between continuity and uniform continuity?

Exercise 4.10

Let $f : (0, 1) \to \mathbb{R}$ be defined by $f(x) = \dfrac{1}{x}$. Show that f is not uniformly continuous on $(0, 1)$.

Task 4.11

Let $f : [a, b] \to \mathbb{R}$ *be a continuous function on the closed interval* $[a, b]$*. Show that* f *is uniformly continuous on* $[a, b]$*.*

Task 4.11

Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$. Show that f is uniformly continuous on $[a, b]$.

Task 4.12

Let $f : D \to \mathbb{R}$ be uniformly continuous on D. If D is a bounded subset of $\mathbb{R}$, then $f(D)$ is also bounded.

Exercise 4.13

Let $f : D \to \mathbb{R}$ be a Lipschitz function on D. Then f is uniformly continuous on D.

Exercise 4.13

Let $f : D \to \mathbb{R}$ be a Lipschitz function on D. Then f is uniformly continuous on D.

Task 4.14

Show: The function $f(x) = \sqrt{x}$ is uniformly continuous on the interval $[0, 1]$, but it is not a Lipschitz function on the interval $[0, 1]$.

Exercise 4.15

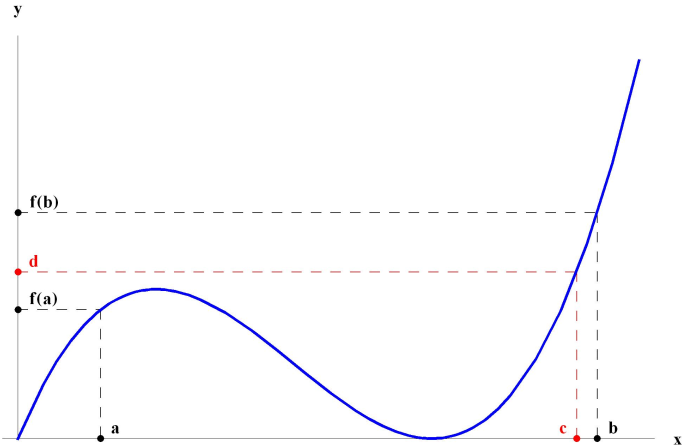
Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$. Then f is bounded on $[a, b]$.

Task 4.16

Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$. Then f has an absolute maximum (and an absolute minimum) on $[a, b]$.

Task 4.17 (Intermediate Value Theorem)

Let $f : I \to \mathbb{R}$ be a continuous function on the interval I. Let $a, b \in I$. If $d \in (f(a), f(b))$, then there is a real number $c \in (a, b)$ such that $f(c) = d$.


y
f(b)
d
f(a)
a
c
b
x

Task 4.18

Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$. Then $f([a, b]) := \{f(x) \mid x \in [a, b]\}$ is also a closed bounded interval.

Task 4.19

Let $f : [a, b] \to \mathbb{R}$ be strictly increasing (or decreasing, resp.) and continuous on $[a, b]$. Show that f has an inverse on $f([a, b])$, which is strictly increasing (or decreasing, resp.) and continuous.

Task 4.19

Let $f : [a, b] \to \mathbb{R}$ be strictly increasing (or decreasing, resp.) and continuous on $[a, b]$. Show that f has an inverse on $f([a, b])$, which is strictly increasing (or decreasing, resp.) and continuous.

Task 4.20

Show that $\sqrt{x} : [0, \infty) \to \mathbb{R}$ is continuous on $[0, \infty)$.

Definition

Let D be a set of real numbers and let $x_0 \in D$ be an accumulation point of D. The function $f : D \to \mathbb{R}$ is said to be DIFFERENTIABLE at x_0, if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exists.}$$

In this case, we call the limit above the DERIVATIVE of f at x_0 and write

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Exercise 5.1

Use the definition to show that $\sqrt[3]{x} : \mathbb{R} \to \mathbb{R}$ *is differentiable at* $x_0 = -27$ *and that its derivative at* $x_0 = -27$ *equals* $\dfrac{1}{27}$.

Exercise 5.1

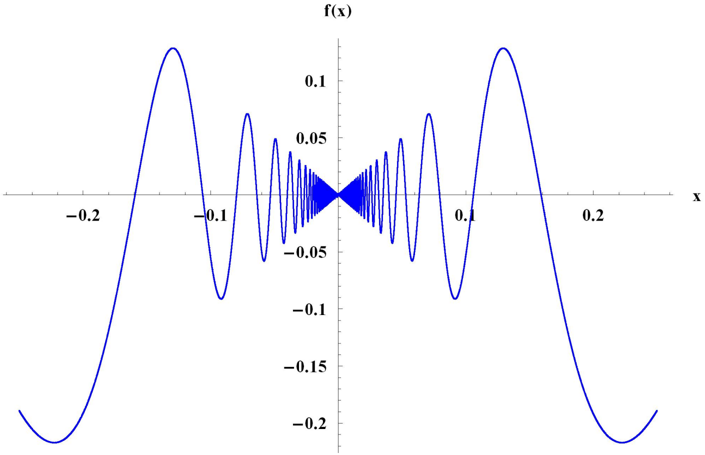
Use the definition to show that $\sqrt[3]{x} : \mathbb{R} \to \mathbb{R}$ *is differentiable at* $x_0 = -27$ *and that its derivative at* $x_0 = -27$ *equals* $\dfrac{1}{27}$.

Exercise 5.2

Let $f : \mathbb{R} \to \mathbb{R}$ *be defined by*

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0,\ x \in \mathbb{R} \\ 0, & \text{if } x = 0 \end{cases}$$

Is $f(x)$ *differentiable at* $x_0 = 0$*?*

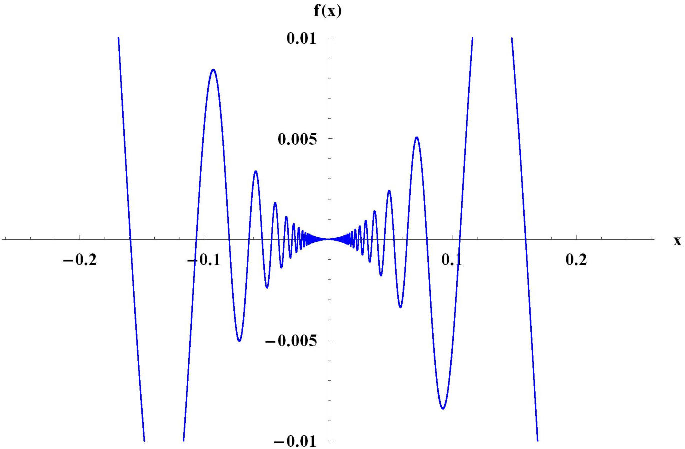

f(x)
0.1
0.05
−0.05
−0.1
−0.15
−0.2
x
−0.2
−0.1
0.1
0.2

Exercise 5.3

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0,\ x \in \mathbb{R} \\ 0, & \text{if } x = 0 \end{cases}$$

Is $f(x)$ differentiable at $x_0 = 0$? Using your Calculus knowledge, compute the derivative at points $x_0 \neq 0$. Is the derivative continuous at $x_0 = 0$?


f(x)
0.01
0.005
−0.005
−0.01
x
−0.2
−0.1
0.1
0.2

Exercise 5.4

Suppose $f : D \to \mathbb{R}$ *is differentiable at* $x_0 \in D$*. Show that* f *is continuous at* x_0*.*

Exercise 5.4

Suppose $f : D \to \mathbb{R}$ *is differentiable at* $x_0 \in D$*. Show that* f *is continuous at* x_0*.*

Exercise 5.5

Give an example of a function with a point at which f *is continuous, but not differentiable.*

Exercise 5.6

Let $f, g : D \to \mathbb{R}$ be differentiable at $x_0 \in D$. Then the function $f + g$ is differentiable at x_0, with $(f+g)'(x_0) = f'(x_0) + g'(x_0)$.

Exercise 5.6

Let $f, g : D \to \mathbb{R}$ be differentiable at $x_0 \in D$. Then the function $f + g$ is differentiable at x_0, with $(f+g)'(x_0) = f'(x_0) + g'(x_0)$.

Task 5.7

Let $f, g : D \to \mathbb{R}$ be differentiable at $x_0 \in D$. Then the function $f \cdot g$ is differentiable at x_0, with

$$(f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0).$$

Exercise 5.8

Show that polynomials are differentiable everywhere. Compute the derivative of a polynomial of the form

$$P(x) = \sum_{k=0}^{n} a_k x^k.$$

Task 5.9

State and prove the "Quotient Rule".

Task 5.9

State and prove the "Quotient Rule".

Task 5.10

State and prove the "Chain Rule".

Task 5.11

Suppose $f : [a, b] \to \mathbb{R}$ *has either a local maximum or a local minimum at* $x_0 \in (a, b)$. *If* f *is differentiable at* x_0, *then* $f'(x_0) = 0$.

Task 5.11

Suppose $f : [a, b] \to \mathbb{R}$ has either a local maximum or a local minimum at $x_0 \in (a, b)$. If f is differentiable at x_0, then $f'(x_0) = 0$.

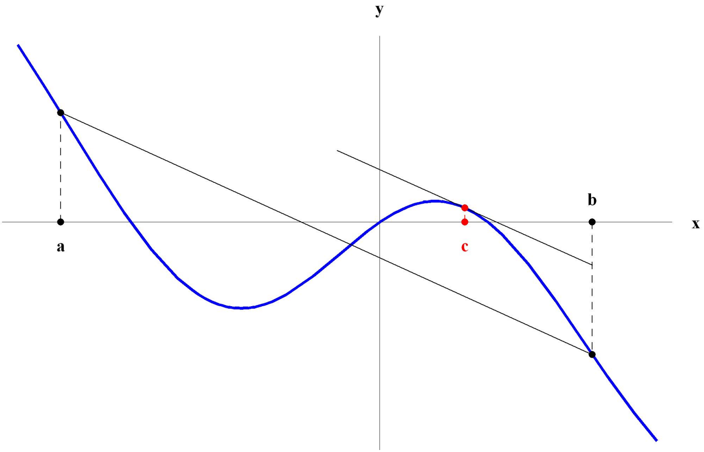
Task 5.12

Suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b).
If $f(a) = f(b) = 0$, then there exists a $c \in (a, b)$ with $f'(c) = 0$.

Task 5.13

Suppose $f : [a, b] \to \mathbb{R}$ *is continuous on* $[a, b]$ *and differentiable on* (a, b).
Then there exists a $c \in (a, b)$ *such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$


y
x
a
b
c

Exercise 5.14

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b).
If $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing.

Exercise 5.15

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b).
If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.

Exercise 5.16

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b).
If $f'(x) \neq 0$ for all $x \in (a, b)$, then f is injective.

Task 5.17

Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ such that $f'(x) \neq 0$ for all $x \in [a, b]$.
Then f is injective; its inverse f^{-1} is differentiable on $f([a, b])$.
Moreover, setting $y = f(x)$, we have

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)}.$$