3 Limits

3.1 Definition and Examples

Let $D \subseteq \mathbb{R}$, let $f : D \to \mathbb{R}$ be a function and let x_0 be an accumulation point of D.

We say that the LIMIT of $f(x)$ at x_0 is equal to $L \in \mathbb{R}$, if for all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon$$

whenever $x \in D$ and $0 < |x - x_0| < \delta$.

In this case we write $\lim\limits_{x \to x_0} f(x) = L$.

Note that—by design—the existence of the limit (and L itself) does not depend on what happens when $x = x_0$, but only on what happens "close" to x_0.

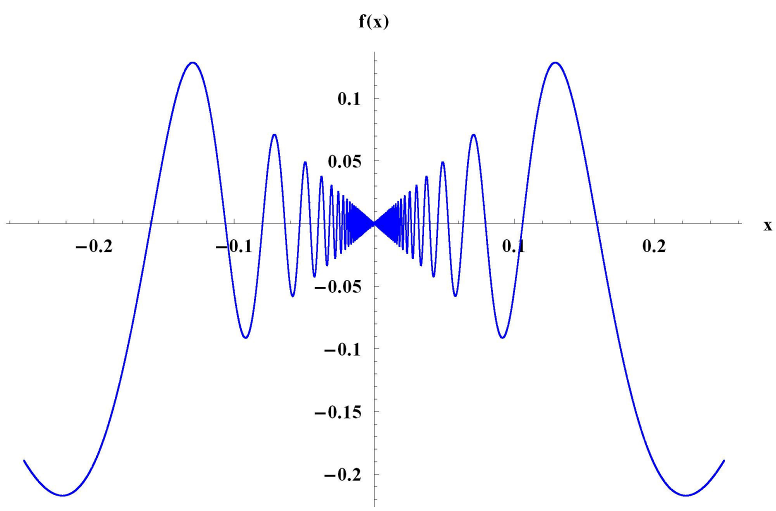

Figure 3: The graph of $x \sin(1/x)$

Exercise 3.1
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0,\ x \in \mathbb{R} \\ 0, & \text{if } x = 0 \end{cases}$$

Does $f(x)$ have a limit a $x_0 = 0$? If so, what is the limit? See Figure 3.

The next two problems reduce the study of the concept of a limit of a function at a point to our earlier study of sequence convergence.

Exercise 3.2

Let $D \subseteq \mathbb{R}$, let $f : D \to \mathbb{R}$ be a function and let x_0 be an accumulation point of D. Then the following are equivalent:

1. $\lim_{x \to x_0} f(x)$ exists and is equal to L.
2. Let (x_n) be any sequence of elements in D that converges to x_0, and satisfies that $x_n \neq x_0$ for all $n \in \mathbb{N}$. Then the sequence $f(x_n)$ converges to L.

Exercise 3.3

What strategy does Exercise 3.2 suggest to show non-existence of a limit at a point?

Without specifying a particular limit, Exercise 3.2 can be phrased as follows:

Optional Task 3.1

Let $D \subseteq \mathbb{R}$, let $f : D \to \mathbb{R}$ be a function and let x_0 be an accumulation point of D. Then the following are equivalent[6]:

1. $\lim_{x \to x_0} f(x)$ exists.
2. Let (x_n) be any sequence of elements in D that converges to x_0, and satisfies that $x_n \neq x_0$ for all $n \in \mathbb{N}$. Then the sequence $f(x_n)$ converges.

There is another way to show existence of a limit without a priori knowledge of what the limit is:

As always, let $D \subseteq \mathbb{R}$, let $f : D \to \mathbb{R}$ be a function and let x_0 be an accumulation point of D. We say that the $f(x)$ has ZERO-OSCILLATION AROUND x_0, if for all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon$$

whenever $x, y \in D$, $0 < |x - x_0| < \delta$, and $0 < |y - x_0| < \delta$.

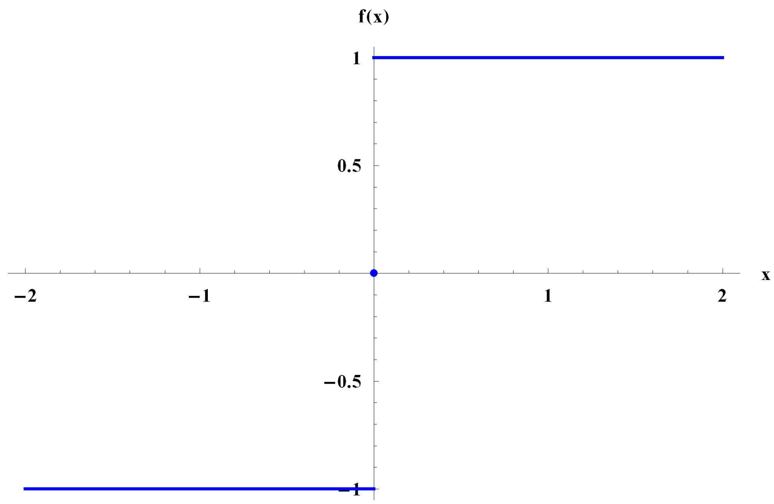

Figure 4: The graph of the function in Exercise 3.4

Optional Task 3.2
Let $D \subseteq \mathbb{R}$, let $f : D \to \mathbb{R}$ be a function and let x_0 be an accumulation point of D. Show that $f(x)$ has a limit at x_0 if and only if $f(x)$ has zero-oscillation around x_0.

Exercise 3.4
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} |x|/x, & \text{if } x \neq 0,\ x \in \mathbb{R} \\ 0, & \text{if } x = 0 \end{cases}$$

Does $f(x)$ have a limit a $x_0 = 0$? If so, what is the limit? See Figure 4.

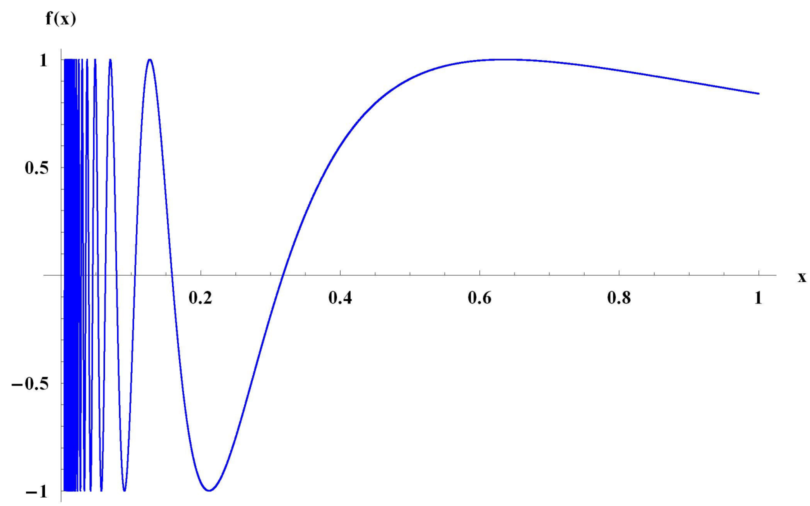

Figure 5: The graph of $\sin(1/x)$

Exercise 3.5
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0,\ x \in \mathbb{R} \\ 0, & \text{if } x = 0 \end{cases}$$

Does $f(x)$ have a limit a $x_0 = 0$? If so, what is the limit? See Figure 5.

Exercise 3.6
Let $f : (0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

For which values of x_0 does $f(x)$ have a limit a x_0? What is the limit?

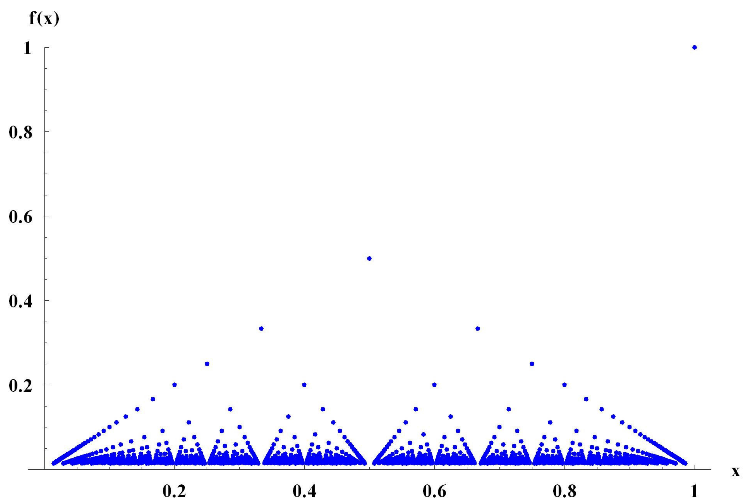

Figure 6: The graph of the function in Task 3.7

Task 3.7
Let $f : (0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \dfrac{1}{q}, & \text{if } x = \dfrac{p}{q} \text{ with } p, q \text{ relatively prime positive integers} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

For which values of x_0 does $f(x)$ have a limit a x_0? What is the limit[7]? See Figure 6.

The result below is called the **Principle of Local Boundedness**.

Exercise 3.8
Let $D \subseteq \mathbb{R}$, let $f : D \to \mathbb{R}$ be a function and let x_0 be an accumulation point of D.

If $f(x)$ has a limit at x_0, then there is a $\delta > 0$ and an $M > 0$ such that

$$|f(x)| \leq M \text{ for all } x \in (x_0 - \delta, x_0 + \delta) \cap D.$$

3.2 Arithmetic of Limits*

Optional Task 3.3
Let $D \subseteq \mathbb{R}$, let $f, g : D \to \mathbb{R}$ be functions and let x_0 be an accumulation point of D.

If $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$, then the sum $f + g$ has a limit at x_0, and $\lim_{x \to x_0} (f + g)(x) = L + M$.

Optional Task 3.4
Let $D \subseteq \mathbb{R}$, let $f, g : D \to \mathbb{R}$ be functions and let x_0 be an accumulation point of D.

If $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$, then the product $f \cdot g$ has a limit at x_0, and $\lim_{x \to x_0} (f \cdot g)(x) = L \cdot M$.

Optional Task 3.5
Let $D \subseteq \mathbb{R}$, let $f : D \to \mathbb{R}$ be a function and let x_0 be an accumulation point of D. Assume additionally that $f(x) \neq 0$ for all $x \in D$.

If $\lim_{x \to x_0} f(x) = L$ and if $L \neq 0$, then the reciprocal function $1/f : D \to \mathbb{R}$ has a limit at x_0, and $\lim_{x \to x_0} \frac{1}{f(x)} = \frac{1}{L}$.

3.3 Monotone Functions*

Let $a < b$ be real numbers. A function $f : [a, b] \to \mathbb{R}$ is called INCREASING on $[a, b]$, if $x < y$ implies $f(x) \leq f(y)$ for all $x, y \in [a, b]$. It is called STRICTLY INCREASING on $[a, b]$, if $x < y$ implies $f(x) < f(y)$ for all $x, y \in [a, b]$.

Similarly, a function $f : [a, b] \to \mathbb{R}$ is called DECREASING on $[a, b]$, if $x < y$ implies $f(x) \geq f(y)$ for all $x, y \in [a, b]$. It is called STRICTLY DECREASING on $[a, b]$, if $x < y$ implies $f(x) > f(y)$ for all $x, y \in [a, b]$.

A function $f : [a, b] \to \mathbb{R}$ is called MONOTONE on $[a, b]$ if it is increasing on $[a, b]$ or it is decreasing on $[a, b]$.

As we have seen in the last section, a function can fail to have limits for various reasons. Monotone functions, on the other hand, are easier to understand: a monotone function fails to have a limit at a point if and only if it "jumps" at that point. The next task makes this precise.

Optional Task 3.6

Let $a < b$ be real numbers and let $f : [a, b]$ be an **increasing** function. Let $x_0 \in (a, b)$. We define

$$L(x_0) = \sup\{f(y) \mid y \in [a, x_0)\}$$

and

$$U(x_0) = \inf\{f(y) \mid y \in (x_0, b]\}$$

Then $f(x)$ has a limit at x_0 if and only if $U(x_0) = L(x_0)$. In this case

$$U(x_0) = L(x_0) = f(x_0) = \lim_{x \to x_0} f(x).$$

Optional Task 3.7

Under the assumptions of the previous task, state and prove a result discussing the existence of a limit at the endpoints a and b.

Optional Task 3.8

Let $a < b$ be real numbers and let $f : [a, b]$ be an increasing function. Show[8] that the set

$$\{y \in [a, b] \mid f(x) \text{ does not have a limit at } y\}$$

is finite or countable[††].

Let us look at an example of an increasing function with countably many "jumps": Let

[††] A set is called COUNTABLE, if all of its elements can be arranged as a sequence $y_1, y_2, y_3, \ldots$ with $y_i \neq y_j$ for all $i \neq j$.

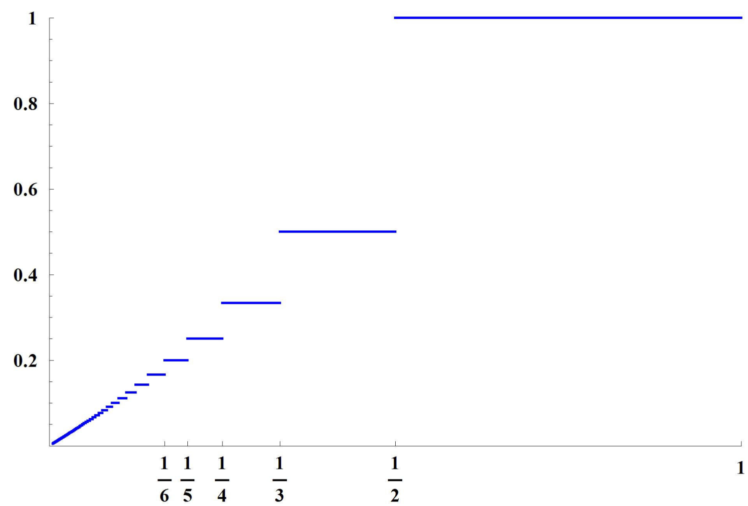

Figure 7: The graph of a function with countable many "jumps"

$g : [0,1] \longrightarrow [0,1]$ be defined as follows:

$$g(x) = \begin{cases} 0 & , \text{ if } x = 0 \\ \dfrac{1}{n} & , \text{ if } x \in \left(\dfrac{1}{n+1}, \dfrac{1}{n} \right] \text{ for some } n \in \mathbb{N} \end{cases}$$

Figure 7 shows the graph of $g(x)$. Note that the function is well defined, since

$$\bigcup_{n \in \mathbb{N}} \left(\frac{1}{n+1}, \frac{1}{n} \right] = (0,1],$$

and

$$\left(\frac{1}{m+1}, \frac{1}{m} \right] \cap \left(\frac{1}{n+1}, \frac{1}{n} \right] = \emptyset$$

for all $m, n \in \mathbb{N}$ with $m \neq n$.

Optional Task 3.9

Show the following:

1. The function $g(x)$ defined above fails to have a limit at all points in the set $D := \left\{ \dfrac{1}{n} \mid n \in \mathbb{N} \right\}$.

2. The function $g(x)$ has a limit at all points in the complement $[0, 1] \setminus D$.