5 The Derivative

5.1 Definition and Examples

Let D be a set of real numbers and let $x_0 \in D$ be an accumulation point of D. The function $f : D \to \mathbb{R}$ is said to be DIFFERENTIABLE at x_0, if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exists.}$$

In this case, we call the limit above the DERIVATIVE of f at x_0 and write

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Exercise 5.1
Use the definition above to show that $\sqrt[3]{x} : \mathbb{R} \to \mathbb{R}$ is differentiable at $x_0 = -27$ and that its derivative at $x_0 = -27$ equals $\dfrac{1}{27}$.

Exercise 5.2
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0,\ x \in \mathbb{R} \\ 0, & \text{if } x = 0 \end{cases}$$

Is $f(x)$ differentiable at $x_0 = 0$? See Figure 3 on page 25.

Exercise 5.3
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0,\ x \in \mathbb{R} \\ 0, & \text{if } x = 0 \end{cases}$$

Is $f(x)$ differentiable at $x_0 = 0$? Using your Calculus knowledge, compute the derivative at points $x_0 \neq 0$. Is the derivative continuous at $x_0 = 0$? See Figure 9 on the next page.

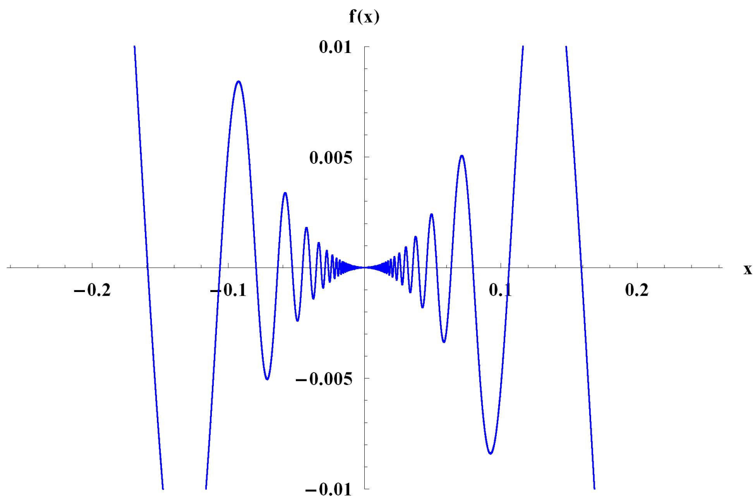

Figure 9: The graph of $x^2 \sin(1/x)$

5.2 Techniques of Differentiation

Exercise 5.4
Suppose $f : D \to \mathbb{R}$ is differentiable at $x_0 \in D$. Show that f is continuous at x_0.

Exercise 5.5
Give an example of a function with a point at which f is continuous, but not differentiable.

Exercise 5.6
Let $f, g : D \to \mathbb{R}$ be differentiable at $x_0 \in D$. Then the function $f+g$ is differentiable at x_0, with $(f + g)'(x_0) = f'(x_0) + g'(x_0)$.

Next come some of the "Calculus Classics", beginning with the "Product Rule":

Task 5.7
Let $f, g : D \to \mathbb{R}$ be differentiable at $x_0 \in D$. Then the function $f \cdot g$ is differentiable at x_0, with

$$(f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0).$$

In particular, if $c \in \mathbb{R}$, then

$$(c \cdot f)'(x_0) = c \cdot f'(x_0).$$

Exercise 5.8
Show that polynomials are differentiable everywhere.

Compute the derivative of a polynomial of the form

$$P(x) = \sum_{k=0}^{n} a_k x^k.$$

Task 5.9
State and prove the "Quotient Rule".

Task 5.10
State and prove the "Chain Rule"[12].

5.3 The Mean-Value Theorem and its Applications

Let D be a subset of $\mathbb{R}$, and let $f : D \to \mathbb{R}$ be a function. We say that f has a LOCAL MAXIMUM at $x_0 \in D$, if there is a neighborhood U of x_0, such that

$$f(x) \leq f(x_0) \text{ for all } x \in U.$$

Similarly, we say that f has a LOCAL MINIMUM at $x_0 \in D$, if there is a neighborhood U of x_0, such that

$$f(x) \geq f(x_0) \text{ for all } x \in U.$$

The next result is commonly known as the **First Derivative Test**[13].

Task 5.11

Suppose $f : [a, b] \to \mathbb{R}$ has either a local maximum or a local minimum at $x_0 \in (a, b)$. If f is differentiable at x_0, then $f'(x_0) = 0$.

Task 5.12

Suppose[14] $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b).

If $f(a) = f(b) = 0$, then there exists a $c \in (a, b)$ with $f'(c) = 0$.

A much more useful version of Task 5.12 is known as the **Mean Value Theorem**:

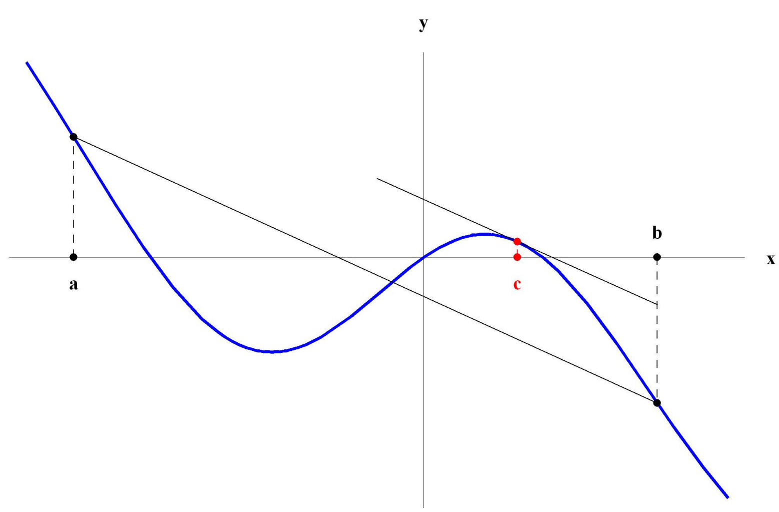

Figure 10: The Mean Value Theorem

Task 5.13

Suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b).

Then there exists a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

See Figure 10 on the page before.

Do not confuse the Mean Value Theorem with the Intermediate Value Theorem!

Nearly all properties of differentiable functions follow from the Mean Value Theorem. The exercises and tasks below are such examples of straightforward applications of the Mean-Value Theorem.

Exercise 5.14

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b).

If $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing.

Exercise 5.15

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b).

If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.

A function $f : D \to \mathbb{R}$ is called INJECTIVE (or 1–1), if $x \neq y$ implies $f(x) \neq f(y)$ for all $x, y \in D$.

Exercise 5.16

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b).

If $f'(x) \neq 0$ for all $x \in (a, b)$, then f is injective.

Task 5.17
Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ such that $f'(x) \neq 0$ for all $x \in [a, b]$.

Then f is injective; its inverse f^{-1} is differentiable on $f([a, b])$. Moreover, setting $y = f(x)$, we have

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)}.$$

5.4 The Derivative and the Intermediate Value Property*

We say that a function $f : [a, b] \to \mathbb{R}$ has the Intermediate Value Property on $[a, b]$ if the following holds: Let $x_1, x_2 \in [a, b]$, and let

$$y \in (f(x_1), f(x_2)).$$

Then there is an $x \in (x_1, x_2)$ satisfying $f(x) = y$.

Recall that we saw earlier that every continuous function has the intermediate value property, see Task 4.17.

On the other hand, not every function with the intermediate value property is continuous:

Optional Task 5.1
Let $f : [-1, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0,\ x \in \mathbb{R} \\ 0, & \text{if } x = 0 \end{cases}$$

Show that f has the intermediate value property on the interval $[-1, 1]$. See Figure 5 on page 28.

The rest of this section will establish the surprising fact that derivatives have the intermediate value property, even though they are not necessarily continuous (see Task 5.3).

Optional Task 5.2
Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$.

If $f'(x) \neq 0$ for all $x \in (a, b)$, then either $f'(x) \geq 0$ for all $x \in [a, b]$ or $f'(x) \leq 0$ for all $x \in [a, b]$.

Optional Task 5.3
Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$. Then $f' : [a, b] \to \mathbb{R}$ has the intermediate value property on $[a, b]$.