# 6 The Integral

**Throughout this chapter all functions are assumed to be bounded.**

## 6.1 Definition and Examples

A finite set $P = \{x_0, x_1, x_2, \ldots, x_n\}$ is called a PARTITION of the interval $[a, b]$, if

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

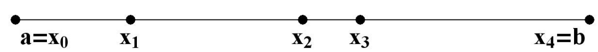


Figure 11: A partition of the interval $[a, b]$.

Let a function $f : [a, b] \to \mathbb{R}$ and a partition $P$ of the interval $[a, b]$ be given. Let $i \in \{1, 2, 3, \ldots, n\}$. We define

$$m_i(f) := \inf\{f(x) \mid x \in [x_{i-1}, x_i]\},$$

and

$$M_i(f) := \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}.$$

The LOWER RIEMANN SUM $\mathcal{L}(f, P)$ is defined as

$$\mathcal{L}(f, P) := \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}).$$

See Figure 12.

The UPPER RIEMANN SUM $\mathcal{U}(f, P)$ is defined as

$$\mathcal{U}(f, P) := \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}).$$

See Figure 13 on the next page.

The LOWER RIEMANN INTEGRAL of $f$ on the interval $[a, b]$ is defined as

$$\mathcal{L}\int_a^b f(x)\,dx := \sup\{\mathcal{L}(f, P) \mid P \text{ is a partition of } [a, b]\}.$$

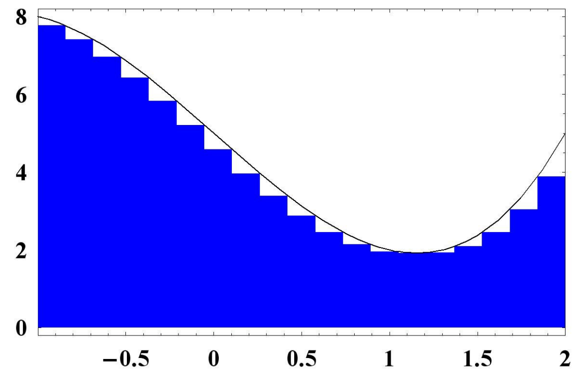


Figure 12: A lower Riemann sum with a partition of 20 equally spaced points.

The UPPER RIEMANN INTEGRAL of $f$ on the interval $[a, b]$ is defined as

$$\mathcal{U}\int_a^b f(x)\,dx := \inf\{\mathcal{U}(f, P) \mid P \text{ is a partition of } [a, b]\}.$$

Let $P$ and $Q$ be two partitions of the interval $[a, b]$. We say that the partition $Q$ is FINER than the partition $P$ if $P \subseteq Q$. In this situation, we also call $P$ COARSER than $Q$.

**Task 6.1**
Let $f : [a, b] \to \mathbb{R}$ be a function, and $P$ and $Q$ be two partitions of the interval $[a, b]$. Assume that $Q$ is finer than $P$. Then

$$\mathcal{L}(f, P) \leq \mathcal{L}(f, Q) \leq \mathcal{U}(f, Q) \leq \mathcal{U}(f, P).$$

Note that Task 6.1 implies that

$$\mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx.$$

We are finally in a position to define the concept of integrability! We will say that a function $f : [a, b] \to \mathbb{R}$ is RIEMANN INTEGRABLE on the interval $[a, b]$, if

$$\mathcal{L}\int_a^b f(x)\,dx = \mathcal{U}\int_a^b f(x)\,dx.$$

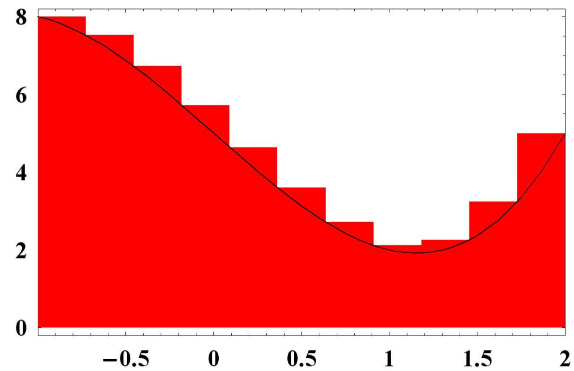


Figure 13: An upper Riemann sum with a partition of 12 equally spaced points.

Their common value is then called the RIEMANN INTEGRAL of $f$ on the interval $[a, b]$ and denoted by

$$\int_a^b f(x)\,dx.$$

**Exercise 6.2**

Use the definition above to compute $\mathcal{L}\int_0^1 x\,dx$ and $\mathcal{U}\int_0^1 x\,dx$. Is the function Riemann integrable on $[0, 1]$?

**Exercise 6.3**

Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Use the definitions above to compute $\mathcal{L}\int_0^1 f(x)\,dx$ and $\mathcal{U}\int_0^1 f(x)\,dx$. Is the function Riemann integrable on $[0, 1]$?

**Task 6.4**
A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if for every $\varepsilon > 0$ there is a partition $P$ of $[a, b]$ such that

$$\mathcal{U}(f, P) - \mathcal{L}(f, P) < \varepsilon.$$

Given a partition $P$, we define its MESH WIDTH $\mu(P)$ as

$$\mu(P) := \max\{x_i - x_{i-1} \mid i = 1, 2, \ldots, n\}.$$

**Task 6.5**
A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if[15] for every $\varepsilon > 0$ there is a $\delta > 0$ such that for **all** partitions $P$ of $[a, b]$ with mesh width $\mu(P) < \delta$

$$\mathcal{U}(f, P) - \mathcal{L}(f, P) < \varepsilon.$$

Two important classes of functions are Riemann-integrable—continuous functions and monotone functions:

**Task 6.6**
If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then $f$ is Riemann integrable on $[a, b]$.

**Task 6.7**
If $f : [a, b] \to \mathbb{R}$ is increasing (or decreasing) on $[a, b]$, then $f$ is Riemann integrable on $[a, b]$.

## 6.2 Arithmetic of Integrals

**Exercise 6.8**
Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$. Then for all $\lambda \in \mathbb{R}$, the function $\lambda f$ is also Riemann integrable on $[a, b]$, and

$$\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx.$$

**Exercise 6.9**
Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$. Then $f + g$ is also Riemann integrable on $[a, b]$, and

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

**Task 6.10**
Let $f : [a, c] \to \mathbb{R}$ be a function and $a < b < c$. Then $f$ is Riemann integrable on $[a, c]$ if and only if $f$ is Riemann integrable on both $[a, b]$ and $[b, c]$. In this case

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

**Exercise 6.11**
Suppose the function $f : [a, b] \to \mathbb{R}$ is bounded above by $M \in \mathbb{R}$: $f(x) \leq M$ for all $x \in [a, b]$. Then

$$\int_a^b f(x)\,dx \leq M \cdot (b - a).$$

## 6.3 The Fundamental Theorem of Calculus

**Task 6.12**
Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there is a number $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

**Task 6.13**
Let $f : [a, b] \to \mathbb{R}$ be bounded and Riemann integrable on $[a, b]$. Let

$$F(x) = \int_a^x f(\tau)\,d\tau.$$

Then $F : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$.

**Task 6.14**
Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Let

$$F(x) = \int_a^x f(\tau)\,d\tau.$$

Then $F : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$, and

$$F'(x) = f(x).$$

The next result is the **Fundamental Theorem of Calculus**.

**Task 6.15**
Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$, and Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$, and suppose $F : [a, b] \to \mathbb{R}$ is an "anti-derivative" of $f(x)$, i.e., $F$ satisfies:

1. $F$ is continuous on $[a, b]$ and differentiable on $(a, b)$,

2. $F'(x) = f(x)$ for all $x \in [a, b]$.

Then

$$\int_a^b f(\tau)\, d\tau = F(b) - F(a).$$