Math 4303 **Worksheet #2** **September 1, 2009**

Suppose that addition and multiplication have already been defined for the set of natural numbers $\mathbb{N}$.

a. We define a relation $\sim$ on $\mathbb{N} \times \mathbb{N}$ as follows:

$$(m, n) \sim (m', n') \Leftrightarrow m + n' = m' + n.$$

Show that $\sim$ defines an equivalence relation on $\mathbb{N} \times \mathbb{N}$.

b. It then makes sense to define equivalence classes $(m, n)_\sim$:

$$(m, n)_\sim := \{(m', n') \in \mathbb{N} \times \mathbb{N} \mid (m', n') \sim (m, n)\}.$$

The set of all these equivalence classes is denoted by $(\mathbb{N} \times \mathbb{N})_\sim$. We will identify the set of integers $\mathbb{Z}$ with this set $(\mathbb{N} \times \mathbb{N})_\sim$.

Which equivalence class corresponds to the integer 0? What about the equivalence classes corresponding to the integers 1 and -3, respectively?

c. How can one define addition of two integers? More precisely, what should be the meaning of

$$(m, n)_\sim + (m', n')_\sim?$$

Is your definition well-defined?

d. Show that addition as defined in c. is commutative.

e. What is the neutral element in $(\mathbb{N} \times \mathbb{N})_\sim$ with respect to addition?

f. Given $(m, n)_\sim \in (\mathbb{N} \times \mathbb{N})_\sim$, what is the inverse element of $(m, n)_\sim$ with respect to addition?

g. How can one define multiplication of two integers? More precisely, what should be the meaning of

$$(m, n)_\sim \cdot (m', n')_\sim?$$

h. Verify that $(2, 5)_\sim \cdot (1, 2)_\sim = (5, 2)_\sim$.