Homework 2 **Analysis** **Fall 2009**

The problems are due on Tuesday, September 22.

For all students:

Problem 1 Let A be a set in a metric space (X, d), and let $x \notin A$. Show that x is an accumulation point of A if and only if there is a sequence (x_n) of elements in A converging to x.

Problem 2 Let A_n, $n \in \mathbb{N}$, be a collection of non-empty subsets of a metric space (M, d) satisfying $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$, and such that $\bigcap_{n=1}^{\infty} A_n = \emptyset$. Show that all points in $\bigcap_{n=1}^{\infty} cl(A_n)$ are accumulation points of A_1.

Problem 3 Let (x_n) and (y_n) be two Cauchy sequences in a metric space (X, d). Show that the sequence $(d(x_n, y_n))$ converges.

Problem 4 Consider the following two binary operations on $(0, 1]$:

$$d(x, y) = |x - y| \text{ for all } x, y \in (0, 1],$$

$$d^*(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right| \text{ for all } x, y \in (0, 1].$$

1. Show that d^* defines a metric on $(0, 1]$.
2. Show that a set is open in the metric space $((0, 1], d)$ if and only if it is open in $((0, 1], d^*)$. Conclude that a sequence converges in the metric space $((0, 1], d)$ if and only if it converges in $((0, 1], d^*)$.
3. Show that the metric space $((0, 1], d^*)$ is complete, while the metric space $((0, 1], d)$ is *not* complete.

Problem 5 Let X be the vector space consisting of all real-valued sequences that are absolutely summing:

$$X = \{(x_n)_{n=1}^{\infty} \mid \sum_{n=1}^{\infty} |x_n| < \infty\}.$$

1. Show that $\|(x_n)_{n=1}^{\infty}\| = \sum_{n=1}^{\infty} |x_n|$ defines a norm on X.
2. For $k \in \mathbb{N}$ define $P_k : X \to \mathbb{R}$ by $P_k((x_n)_{n=1}^{\infty}) = x_k$. Find a sequence $(\underline{\mathbf{x}}_i)_{i=1}^{\infty}$ of elements in X such that $\lim_{i \to \infty} P_k(\underline{\mathbf{x}}_i)$ exists for all $k \in \mathbb{N}$, but such that the sequence $(\underline{\mathbf{x}}_i)_{i=1}^{\infty}$ itself fails to be a Cauchy sequence.