Homework 3 **Analysis** **Fall 2009**

The problems are due on Thursday, October 1.

For all students:

Problem 1 Let (A_n) be a sequence of non-empty closed bounded subsets in a complete metric space (X, d), such that $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$, and such that $\lim_{n\to\infty} diam\, A_n = 0$. Show that $\bigcap_{n=1}^{\infty} A_n$ consists of exactly one point.

The diameter of a bounded set A in a metric space is defined as:

$$diam\, A = \sup\{d(x, y) \mid x, y \in A\}.$$

Problem 2 Show that a set A in a metric space (M, d) is compact (see the definition on p. 152) if and only if (A, d) is a compact metric space.

Problem 3 A metric space (X, d) is called *countably compact*, if every open cover of X, which consists of countably many open sets, contains a finite subcover. Show that every countably compact metric space is sequentially compact.

Problem 4 Consider C, the normed vector space of continuous functions on the interval $[0, 1]$, endowed with the sup-norm $\|f\| = \sup_{x\in[0,1]} |f(x)|$. Show that $B = \{f \in C \mid \|f\| \leq 1\}$ fails to be sequentially compact.

Hint: Find a sequence (f_n) in B which satisfies $\|f_n - f_m\| \geq 1$ for all $n \neq m$.

Problem 5 A metric space is called *separable*, if it contains a countable dense subset. Show that every compact metric space is separable.