The problems are due on Thursday, October 29.

For all students:

Problem 1 Let X and Y be two metric spaces. Show that $f : X \to Y$ is continuous if and only if for all subsets A, B of X with $cl(A) = cl(B)$ in X, their images $f(A)$ and $f(B)$ satisfy $cl(f(A)) = cl(f(B))$ in Y.

Problem 2 Let (X, d) be a compact metric space, and $f : X \to X$ be a map satisfying

$$d(f(x), f(y)) = d(x, y) \text{ for all } x, y \in X.$$

Denote by

$$f^n := \underbrace{f \circ f \ldots \circ f}_{n \text{ times}}.$$

(a) Show that f is 1-1.
(b) Show that f is onto. Proceed as follows: If $y \notin f(X)$, consider the sequence $(f^n(y))$. Show that this sequence contains no converging subsequence.

Problem 3 Let (a_n) be a sequence of real numbers, such that $\sum |a_n|$ converges, and let $\rho : \mathbb{N} \to \mathbb{N}$ be any bijection. Show that

$$\sum_{n=1}^{\infty} a_{\rho(n)} = \sum_{n=1}^{\infty} a_n.$$

Problem 4 For a real number x, define $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$.
Let (a_n) be a sequence of real numbers, such that $\sum a_n$ converges, while $\sum |a_n|$ diverges.

1. Show that the partial sums of $\sum a_n^+$ and the partial sums of $\sum a_n^-$ are unbounded increasing sequences. Note that $\lim_{n\to\infty} a_n^+ = 0$ and $\lim_{n\to\infty} a_n^- = 0$.
2. Let $L \in \mathbb{R}$. Using the results above, show that there is a bijection $\rho : \mathbb{N} \to \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} a_{\rho(n)} = L.$$

For graduate students:

Problem 5 A set A in $\mathbb{R}^n$ is called *nowhere dense*, if the interior of its closure is empty: $int(cl\, A) = \emptyset$. Show that $\mathbb{R}^n$ cannot be written as the countable union of nowhere dense sets.