Homework 6 **Analysis** **Fall 2009**

The problems are due on Thursday, November 12.

For all students:

Problem 1 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as follows:

$$f(x,y) = \begin{cases} 0, & \text{if } (x,y) = (0,0) \\ \dfrac{xy^2}{x^2+y^4}, & \text{if } (x,y) \neq (0,0) \end{cases}$$

1. Show that f is bounded. (This means there is an $M \in \mathbb{R}$ such that $|f(x,y)| \leq M$ for all $(x,y) \in \mathbb{R}^2$.)
2. Let $(x_0, y_0) \neq (0,0)$. Define $\ell : \mathbb{R} \to \mathbb{R}^2$ by $\ell(t) = (tx_0, ty_0)$. Show that $f \circ \ell : \mathbb{R} \to \mathbb{R}$ is continuous at 0. (This means that the function f is continuous at $(0,0)$ "in every direction.")
3. Show that f is not continuous at $(0,0)$.

Problem 2 Let $g : [0,1] \to \mathbb{R}$ be defined as follows:

$$g(x) = \begin{cases} 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \text{ or } x = 0 \\ \dfrac{1}{q}, & \text{if } x \in \mathbb{Q} \setminus \{0\}, \text{ where } x = \frac{p}{q} \text{ with } p, q \in \mathbb{N} \text{ and } \gcd(p,q) = 1 \end{cases}$$

Find all points in $[0,1]$ where g is continuous.

Problem 3 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$. Assume also that $\lim_{|x| \to \infty} f(x) = 0$. (This means that for all $\varepsilon > 0$ there is an $N \in \mathbb{R}$ such that $|f(x)| < \varepsilon$ whenever $|x| > N$.) Show that f is uniformly continuous on $\mathbb{R}$.

Problem 4 Let (X,d) and (Y,ρ) be metric spaces and $D \subset X$.

1. Suppose $f : D \to Y$ is uniformly continuous on D. Show: If (x_n) is a Cauchy sequence in D, then $(f(x_n))$ is a Cauchy sequence in Y.
2. Show that the result above fails if one only assumes that f is continuous on D.

Problem 5 (Extra Credit) Let (X,d) and (Y,ρ) be arbitrary metric spaces. Give a characterization of all the sets $D \subset X$ such that every continuous function $f : D \to Y$ is uniformly continuous on D. (Note that every function $f : \mathbb{N} \to \mathbb{R}$ is uniformly continuous.)