**Homework 7** **Analysis** **Fall 2009**

*The problems are due on Thursday, December 3.*

## For all students:

**Problem 1** Let $K$ be a compact subset of $\mathbb{R}$, and for all $n \in \mathbb{N}$ let $f_n : K \to \mathbb{R}$ be a continuous function. Prove or disprove: If $(f_n)$ converges pointwise to a continuous function $f : K \to \mathbb{R}$, then $(f_n)$ converges uniformly to $f$.

**Problem 2** A function $f : \mathbb{R} \to \mathbb{R}$ is called *lower semi-continuous* if whenever $x \in \mathbb{R}$ and $\alpha \in \mathbb{R}$ satisfy $f(x) > \alpha$, there exists a $\delta > 0$ such that $f(y) > \alpha$ for all $y \in \mathbb{R}$ satisfying $|x - y| < \delta$.

1. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is lower semi-continuous, but not continuous.
2. Show: If a sequence of lower semi-continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ satisfies
   (a) $f_{n+1}(x) \geq f_n(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, and
   (b) $(f_n)$ converges pointwise to a function $f$,

   then $f$ is also lower semi-continuous.

**Problem 3** Let $f : \mathbb{R} \to \mathbb{R}$ be the limit of a uniformly converging sequence of polynomials $P_n : \mathbb{R} \to \mathbb{R}$. Show that $f$ is a polynomial.

## For graduate students:

**Problem 4 (20 points)** Let $(f_n) : \mathbb{R} \to [0, 1]$ be a sequence of *increasing* functions, i.e. satisfying $f_n(x) \leq f_n(y)$ for all $x \leq y$ and all $n \in \mathbb{N}$. The problem will establish that a subsequence of $(f_n)$ converges pointwise on $\mathbb{R}$.

1. Show that there is a subsequence $(f_{n_k})$ of $(f_n)$, which converges at all rational points in $\mathbb{R}$, say $\lim_{k\to\infty} f_{n_k}(q) =: f(q)$ for all rational numbers $q$.
2. For $x \in \mathbb{R}$ define $f(x) := \sup\{f(q) \mid q \leq x\ ,\ q \text{ is rational}\}$. Using the monotonicity of the functions involved, show that $f(x) = \lim_{k\to\infty} f_{n_k}(x)$ for all $x \in \mathbb{R}$, at which $f$ is continuous.
3. Show that a further subsequence of $(f_{n_k})$ converges to $f$ for *all* $x \in \mathbb{R}$.
   *Hint: How many points are there, at which $f$ is discontinuous?*