The assignment is due at the beginning of class on October 6, 2010.

Problem 1 (10 points) A sequence (a_n) is called *proper*, if $a_n \neq a_m$ for all $n \neq m$.

1. Show that a proper bounded sequence (a_n) converges, if $\{a_n \mid n \in \mathbb{N}\}$ has **exactly one** accumulation point.
2. Show that 1. fails if we omit the hypothesis that the sequence is bounded.
3. Show that 1. fails if we omit the hypothesis that the sequence is proper.

Problem 2 (15 points) Let $(a_n)_{n=1}^{\infty}$ be a Cauchy sequence, and let $\varphi : \mathbb{N} \to \mathbb{N}$ be a one-to-one function. Show that the sequence $\left(a_{\varphi(n)}\right)_{n=1}^{\infty}$ is Cauchy.

Problem 3 (15 points) Problem 31 on page 56.

Problem 4 (10 points) This problem outlines the beginning of a *construction* of the real numbers from the rational numbers.

Let us denote the set of all Cauchy sequences of rational numbers by $\mathcal{C}$. We say that two Cauchy sequences (a_n) and (b_n) of rational numbers are *equivalent*, if

$$\lim_{n\to\infty} (a_n - b_n) = 0.$$

If two Cauchy sequences (a_n) and (b_n) are equivalent, we write $(a_n) \sim (b_n)$.

1. Show that $\sim$ is indeed an equivalence relation, i.e, show for all $(a_n), (b_n)$ and $(c_n) \in \mathcal{C}$:
   (a) $(a_n) \sim (a_n)$ (Reflexivity)
   (b) $(a_n) \sim (b_n) \;\Rightarrow\; (b_n) \sim (a_n)$ (Symmetry)
   (c) $(a_n) \sim (b_n)$ and $(b_n) \sim (c_n) \;\Rightarrow\; (a_n) \sim (c_n)$ (Transitivity)

The equivalence class $[(a_n)]$ then is the set of all Cauchy sequences of rational numbers equivalent to (a_n):

$$[(a_n)] := \{(b_n) \in \mathcal{C} \mid (b_n) \sim (a_n)\}$$

Note that $[(a_n)] = [(b_n)]$ if and only if $(a_n) \sim (b_n)$.

We denote the set of all such equivalence classes by $\mathcal{R}$. $\mathcal{R}$ can be considered as a *model* for the set of real numbers $\mathbb{R}$. To every equivalence class in $\mathcal{R}$ there corresponds in a unique way a real number in $\mathbb{R}$: The real number corresponding to $[(a_n)] \in \mathcal{R}$ is its limit $\lim_{n\to\infty} a_n$.

2. Show that under this correspondence an equivalence class $[(a_n)]$ represents a rational number if and only if (a_n) is equivalent to a constant sequence.

There will be a follow-up problem on a later homework sheet.