The assignment is due at the beginning of class on October 18, 2010.

Let (a_n) be a bounded sequence of real numbers. We define two real numbers $L(a_n)$ and $H(a_n)$ as follows:

$$L(a_n) := \lim_{k\to\infty} \left(\inf\{a_n \mid n \geq k\}\right),$$

and

$$H(a_n) := \lim_{k\to\infty} \left(\sup\{a_n \mid n \geq k\}\right).$$

Problem 1 (10 points) Explain why $H(a_n)$ exists for every bounded sequence (a_n). (The same is true for $L(a_n)$.)

Problem 2 (10 points) Show that $L(a_n) = \sup\left\{\inf\{a_n \mid n \geq k\} \mid k \in \mathbb{N}\right\}$.

Problem 3 (10 points) Show that a bounded sequence (a_n) converges if and only if

$$L(a_n) = H(a_n).$$

Problem 4 (10 points) Let (a_n) be a bounded sequence of real numbers, and let (a_{n_k}) be one of its converging subsequences. Show that

$$L(a_n) \leq \lim_{k\to\infty} a_{n_k} \leq H(a_n).$$

Problem 5 (10 points) Let (a_n) be a bounded sequence of real numbers. Show that (a_n) has a subsequence that converges to $H(a_n)$.