The assignment is due at the beginning of class on October 18, 2010.

Let (a_n) be a bounded sequence of real numbers. We define two real numbers $L(a_n)$ and $H(a_n)$ as follows:

$$L(a_n) := \lim_{k\to\infty} (\inf\{a_n \mid n \geq k\}), \text{ and } H(a_n) := \lim_{k\to\infty} (\sup\{a_n \mid n \geq k\}).$$

Problem 1 (10 points) Explain why $H(a_n)$ exists for every bounded sequence (a_n). (The same is true for $L(a_n)$.)

Let $A_k = \{a_n \mid n \geq k\}$ and $s_k = \sup A_k$. (Note that the sets A_k are bounded and non-empty.) Since (A_k) is a decreasing family of sets, (s_k) is a decreasing sequence. Moreover (s_k) is bounded from below, since (a_n) is a bounded sequence. Consequently (s_k) is convergent.

Problem 2 (10 points) Show that $L(a_n) = \sup\left\{\inf\{a_n \mid n \geq k\} \mid k \in \mathbb{N}\right\}$.

If we set $i_k = \inf\{a_n \mid n \geq k\}$, we obtain—in analogy to the first problem—that (i_k) is an increasing bounded sequence. Such sequences converge to their supremum.

Problem 3 (10 points) Show that a bounded sequence (a_n) converges if and only if $L(a_n) = H(a_n)$.

Suppose the sequence (a_n) converges to ℓ, and let $\varepsilon > 0$. Then there is a $K \in \mathbb{N}$ such $\ell - \varepsilon < a_n < \ell + \varepsilon$ for all $n \geq K$. Thus $s_K \leq \ell + \varepsilon$ and $i_K \geq \ell - \varepsilon$, and consequently $H(a_n) \leq \ell + \varepsilon$ and $L(a_n) \geq \ell - \varepsilon$. Thus $H(a_n) - L(a_n) \leq 2\varepsilon$. But it follows directly from the definitions above that $L(a_n) \leq H(a_n)$. Thus $|H(a_n) - L(a_n)| \leq 2\varepsilon$. Since this estimate holds for all $\varepsilon > 0$, we have shown that $L(a_n) = H(a_n)$.

On the other hand, for a given $\varepsilon > 0$, we can find a $K \in \mathbb{N}$ such that $|i_K - L(a_n)| < \varepsilon/2$ and $|s_K - H(a_n)| < \varepsilon/2$. So if $L(a_n) = H(a_n)$, we obtain that $0 \leq s_K - i_K < \varepsilon$. Thus for any $m, n \geq K$, $|a_m - a_n| \leq \varepsilon$, and consequently (a_n) is a Cauchy sequence.

Problem 4 (10 points) Let (a_n) be a bounded sequence of real numbers, and let (a_{n_k}) be one of its converging subsequences. Show that

$$L(a_n) \leq \lim_{k\to\infty} a_{n_k} \leq H(a_n).$$

For any $K \in \mathbb{N}$, $\{a_{n_k} \mid k \geq K\} \subseteq \{a_n \mid n \geq K\}$. Consequently $L(a_{n_k}) \geq L(a_n)$ and $H(a_{n_k}) \leq H(a_n)$. The result follows now from Problem 3 and the earlier observed fact that $L(x_n) \leq H(x_n)$ for any bounded sequence (x_n).

Problem 5 (10 points) Let (a_n) be a bounded sequence of real numbers. Show that (a_n) has a subsequence that converges to $H(a_n)$.

Let $\varepsilon = 1$. Since (s_k) converges to $H(a_n)$, we can find a $K \in \mathbb{N}$ such that $0 \leq s_K - H(a_n) \leq \varepsilon$. By the definition of s_K as a supremum, we can then pick $n_1 > K$ such that $|a_{n_1} - H(a_n)| \leq \varepsilon$. Once $n_1, n_2, \ldots, n_{k-1}$ have been chosen, we let $\varepsilon = 1/k$. As before we can find a $K > n_{k-1}$ such that $0 \leq s_K - H(a_n) \leq \varepsilon$, and then pick $n_k > K > n_{k-1}$ such that $|a_{n_k} - H(a_n)| \leq \varepsilon$. By our construction, $|a_{n_k} - H(a_n)| \leq 1/k$ for all $k \in \mathbb{N}$; so (a_{n_k}) converges to $H(a_n)$.