Math 3341 **Homework 5** **Fall 2010**

The assignment is due at the beginning of class on November 1, 2010.

Problem 1 (15 points) Let $f : [a, b] \to \mathbb{R}$ be an increasing function. Show that $\lim_{x\to a} f(x)$ exists. What can you say about the relationship between the limit and $f(a)$?

The last three problems require the following setup: Let $\{q_k \mid k \in \mathbb{N}\}$ be a fixed enumeration of $\mathbb{Q} \cap (0, 1)$. For $k \in \mathbb{N}$ define $f_k : [0, 1] \to \mathbb{R}$ by

$$f_k(x) = \begin{cases} 0, & \text{if } x < q_k \\ 2^{-k}, & \text{if } x \geq q_k \end{cases}$$

We then define $s_n : [0, 1] \to \mathbb{R}$ by $s_n = f_1 + f_2 + f_3 + \cdots + f_n$.

Fix $x \in [0, 1]$. Since $f_k(x) \geq 0$ for all $k \in \mathbb{N}$, the sequence $(s_n(x))_{n=1}^{\infty}$ is increasing. Additionally $0 \leq s_n(x) \leq 1$ for all $n \in \mathbb{N}$. Therefore the sequence $(s_n(x))_{n=1}^{\infty}$ converges.

We will set $s(x) = \lim_{n\to\infty} s_n(x)$. This defines a function $s : [0, 1] \to \mathbb{R}$.

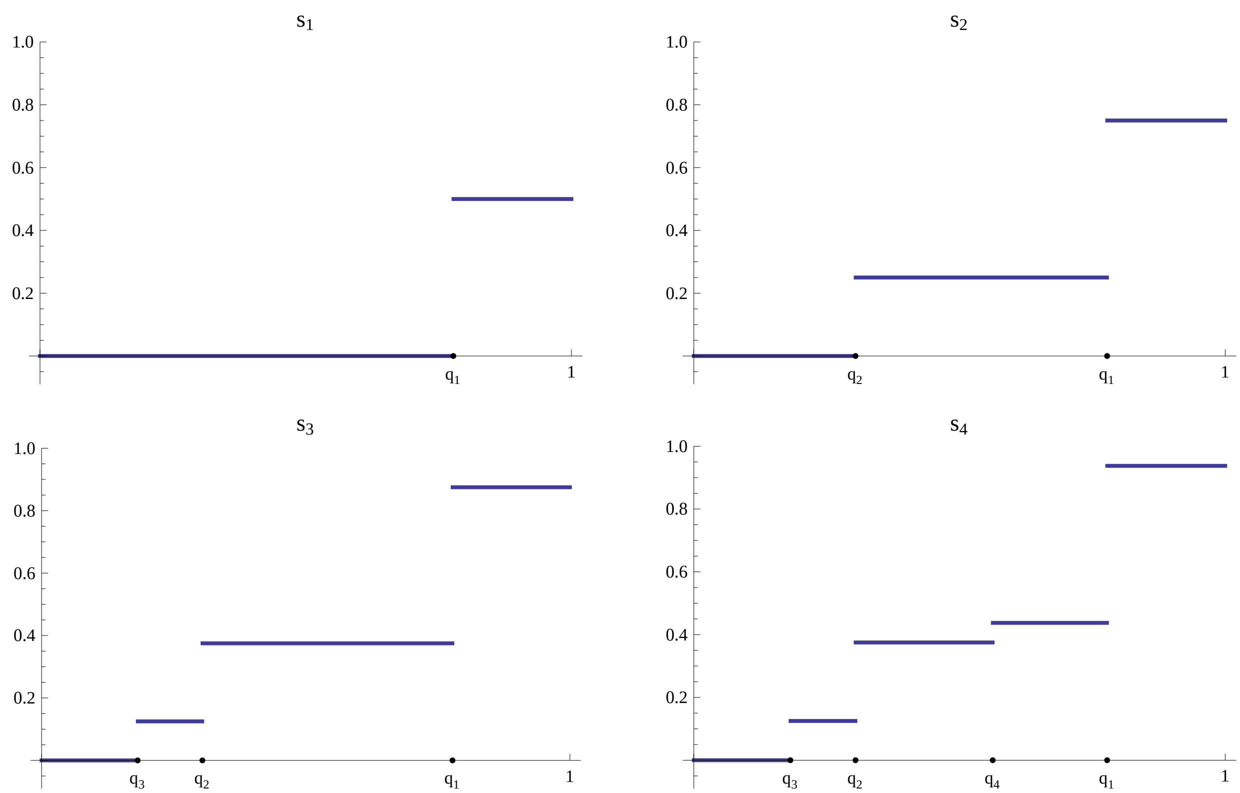

Figure 1: Graphs of $s_1, \ldots, s_4$ for a fixed enumeration $\{q_k \mid k \in \mathbb{N}\}$

Problem 2 (10 points) Show that the function $s : [0, 1] \to \mathbb{R}$ is increasing. Show that $s(0) = 0$ and $s(1) = 1$.

Problem 3 (15 points) Let $k \in \mathbb{N}$. Show that $\lim_{x\to q_k} s(x)$ does not exist.

Problem 4 (10 points) Let x_0 be irrational in $[0, 1]$. Show that $\lim_{x\to x_0} s(x)$ exists.