and the statement holds for $n = r + 1$. By induction, the statement holds for all n, and $\{a_n\}_{n=1}^{\infty}$ converges to 2. ■

■ ***Example 1.10*** Recall now a word of philosophy mentioned earlier—namely, that determining the limit of a sequence may be half the battle in showing a sequence to be convergent. Consider the sequence $\{\sqrt[n]{n}\}_{n=1}^{\infty}$. Let us try to guess the limit in advance by reasoning similar to that used in the preceding paragraph. Suppose $\{\sqrt[n]{n}\}_{n=1}^{\infty}$ converges; call the limit L. Now consider the subsequence $\{\sqrt[2n]{2n}\}_{n=1}^{\infty}$;

$$\sqrt[2n]{2n} = \sqrt{\sqrt[n]{2}\sqrt[n]{n}},$$

and we know $\{\sqrt[n]{2}\}_{n=1}^{\infty}$ converges to 1 (see Exercise 38). Thus, $\{\sqrt[2n]{2n}\}_{n=1}^{\infty}$ converges to L and also to $\sqrt{L}$; hence, by arguments given before, $L = 1$. We shall try to prove that the sequence converges to 1 or, equivalently, that the sequence $\{\sqrt[n]{n} - 1\}_{n=1}^{\infty}$ converges to 0. Let $x_n = \sqrt[n]{n} - 1$; clearly $x_n \geq 0$ and

$$n = (1 + x_n)^n = 1 + nx_n + \frac{n(n-1)}{2}x_n^2 + \cdots + x_n^n \geq \frac{n(n-1)}{2}x_n^2.$$

Thus, for all $n \geq 2$, we have $0 \leq x_n \leq \sqrt{2/(n-1)}$. It should be clear now how to complete the proof that $\{x_n\}_{n=1}^{\infty}$ converges to 0. ■

EXERCISES

1.1 SEQUENCES AND CONVERGENCE

1. Show that $[0, 1]$ is a neighborhood of $\frac{2}{3}$—that is, there is $\epsilon > 0$ such that

$$\left(\frac{2}{3} - \epsilon, \frac{2}{3} + \epsilon\right) \subset [0, 1].$$

***2.** Let x and y be distinct real numbers. Prove there is a neighborhood P of x and a neighborhood Q of y such that $P \cap Q = \varnothing$.

***3.** Suppose x is a real number and $\epsilon > 0$. Prove that $(x - \epsilon, x + \epsilon)$ is a neighborhood of each of its members; in other words, if $y \in (x - \epsilon, x + \epsilon)$, then there is $\delta > 0$ such that $(y - \delta, y + \delta) \subset (x - \epsilon, x + \epsilon)$.

4. Find upper and lower bounds for the sequence $\left\{\frac{3n+7}{n}\right\}_{n=1}^{\infty}$.

5. Give an example of a sequence that is bounded but not convergent.

6. Use the definition of convergence to prove that each of the following sequences converges:

(a) $\left\{5 + \frac{1}{n}\right\}_{n=1}^{\infty}$

(b) $\left\{\frac{2 - 2n}{n}\right\}_{n=1}^{\infty}$