Math 3341 **Homework 2** **Spring 2011**

The assignment is due at the beginning of class on February 17, 2011.

Problem 1 (10 points) Show from first principles that every Cauchy sequence is bounded.

Problem 2 (15 points) Let $\mathcal{A}(S)$ denote the set of accumulation points of a set S.
Suppose S and T are sets of real numbers.

1. Show: If $S \subseteq T$, then $\mathcal{A}(S) \subseteq \mathcal{A}(T)$.
2. Is it true that $\mathcal{A}(S \cup T) = \mathcal{A}(S) \cup \mathcal{A}(T)$? (Give a proof or provide a counterexample.)
3. Is it true that $\mathcal{A}(S \cap T) = \mathcal{A}(S) \cap \mathcal{A}(T)$? (Give a proof or provide a counterexample.)

Problem 3 (10 points) Find the set of accumulation points of the set $\left\{ \dfrac{1}{m} + \dfrac{1}{n} \mid m, n \in \mathbb{N} \right\}$.

Problem 4 (15 points) Let us denote the set of all Cauchy sequences of rational numbers by $\mathcal{C}$.
We say that two Cauchy sequences (a_n) and (b_n) of rational numbers are *equivalent* (written as $(a_n) \sim (b_n)$), if

$$\lim_{n\to\infty} (a_n - b_n) = 0.$$

We say $(a_n) \preceq (b_n)$ if for all $k \in \mathbb{N}$ there is an $N \in \mathbb{N}$ such that $a_n \leq b_n + \frac{1}{k}$ for all $n \geq N$.

1. Show that $\sim$ is indeed an equivalence relation, i.e, show for all $(a_n), (b_n)$ and $(c_n) \in \mathcal{C}$:
   (a) $(a_n) \sim (a_n)$ (Reflexivity)
   (b) $(a_n) \sim (b_n) \;\Rightarrow\; (b_n) \sim (a_n)$ (Symmetry)
   (c) $(a_n) \sim (b_n)$ and $(b_n) \sim (c_n) \;\Rightarrow\; (a_n) \sim (c_n)$ (Transitivity)
2. Show that $\preceq$ is transitive on $\mathcal{C}$.
3. Show for all $(a_n), (b_n) \in \mathcal{C}$: If $(a_n) \preceq (b_n)$ and $(b_n) \preceq (a_n)$, then $(a_n) \sim (b_n)$.