

*The assignment is due at the beginning of class on April 7, 2011.*

**Problem 1 (10 points)** Let  $f, g : D \rightarrow \mathbb{R}$  be two **bounded** functions that are uniformly continuous on  $D$ . Show that their product  $f \cdot g$  is uniformly continuous on  $D$ .

**Problem 2 (15 points)** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous on  $\mathbb{R}$ , and assume that for all  $\varepsilon > 0$  there is an  $N > 0$  such that  $|f(x)| < \varepsilon$  for all  $x$  satisfying  $|x| > N$ . Show that  $f$  is uniformly continuous on  $\mathbb{R}$ .

**Problem 3 (15 points)** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a function. We say  $f$  satisfies  $(*)$  if there is an  $M > 0$  such that  $|f(x) - f(y)| \leq M \cdot |x - y|$  for all  $x, y \in [a, b]$ .

1. Show: If  $f$  satisfies  $(*)$ , then  $f$  is uniformly continuous on  $[a, b]$ .
2. Let  $f : [0, 1] \rightarrow \mathbb{R}$  be given by  $f(x) = \sqrt{x}$ . Show that  $f$  does not satisfy  $(*)$ .

**Problem 4 (10 points)** Let  $S$  be a set of real numbers satisfying the following property: Every sequence in  $S$  has a subsequence that converges to an element in  $S$ .

1. Show that  $S$  is bounded.
2. Show that  $S$  is closed.