Math 3341 **Homework 7** **Fall 2011**

The assignment is due at the beginning of class on November 30.

Problem 1 (10 points) Let $f : [a,b] \to \mathbb{R}$ be an increasing function, and let $x_0 \in (a,b)$. Let $L = \sup\{f(x) \mid a \le x < x_0\}$ and $U = \inf\{f(x) \mid x_0 < x \le b\}$. Show that f is continuous at x_0 if and only if $L = U$.

Problem 2 (10 points) Let $f : [a,b] \to \mathbb{R}$ be an increasing function. Show that $\lim\limits_{x\to a} f(x)$ exists. What can you say about the relationship between this limit and $f(a)$?

The last three problems require the following setup: Let $\{q_k \mid k \in \mathbb{N}\}$ be a fixed enumeration of $\mathbb{Q} \cap (0,1)$. For $k \in \mathbb{N}$ define $f_k : [0,1] \to \mathbb{R}$ by

$$f_k(x) = \begin{cases} 0, & \text{if } x < q_k \\ 2^{-k}, & \text{if } x \ge q_k \end{cases}$$

We then define $s_n : [0,1] \to \mathbb{R}$ by $s_n = f_1 + f_2 + f_3 + \cdots + f_n$.

Fix $x \in [0,1]$. Since $f_k(x) \ge 0$ for all $k \in \mathbb{N}$, the sequence $(s_n(x))_{n=1}^{\infty}$ is increasing. Additionally $0 \le s_n(x) \le 1$ for all $n \in \mathbb{N}$. Therefore the sequence $(s_n(x))_{n=1}^{\infty}$ converges.

We will set $s(x) = \lim\limits_{n\to\infty} s_n(x)$. This defines a function $s : [0,1] \to \mathbb{R}$.

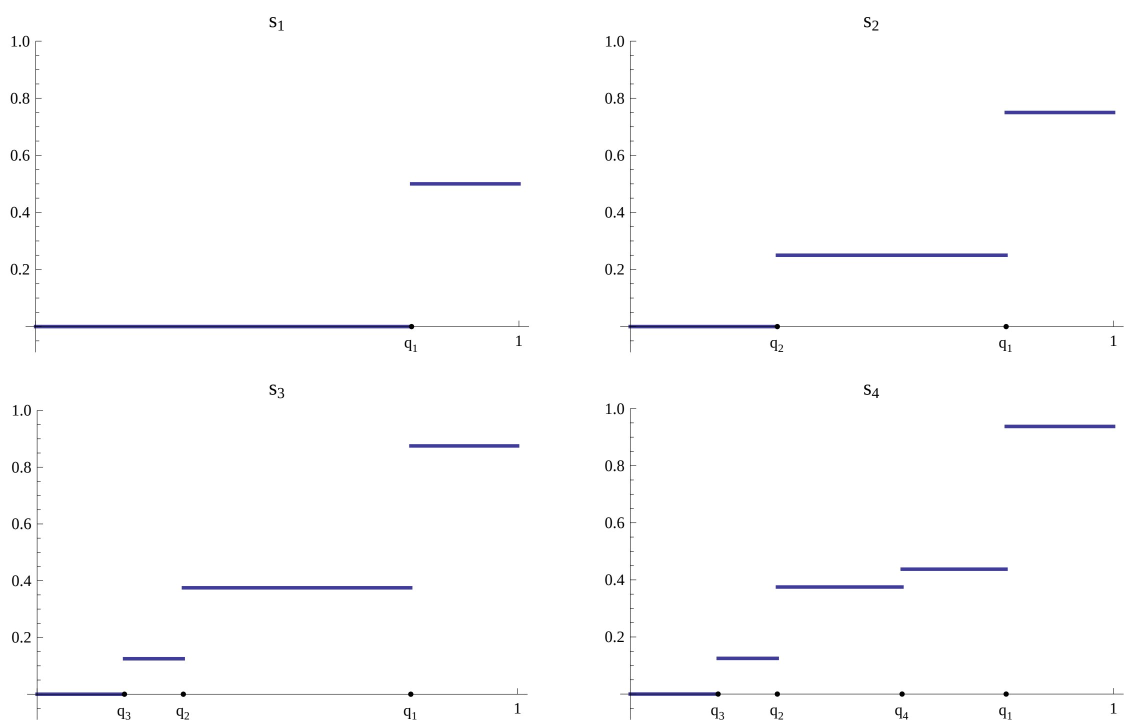

Figure 1: Graphs of $s_1, \dots, s_4$ for a fixed enumeration $\{q_k \mid k \in \mathbb{N}\}$

Problem 3 (10 points) Show that the function $s : [0,1] \to \mathbb{R}$ is increasing. Show that $s(0) = 0$ and $s(1) = 1$.

Problem 4 (10 points) Let $k \in \mathbb{N}$. Show that $\lim\limits_{x\to q_k} s(x)$ does not exist.

Problem 5 (10 points) Let x_0 be irrational in $[0,1]$. Show that $\lim\limits_{x\to x_0} s(x)$ exists.