The fundamental theorem of algebra

Chapter 19

Every nonconstant polynomial with complex coefficients has at least one root in the field of complex numbers.

It has been commented upon that the "Fundamental theorem of algebra" is not really fundamental, that it is not necessarily a theorem since sometimes it serves as a definition, and that in its classical form it is not a result from algebra, but rather from analysis.

Gauss called this theorem, for which he gave seven proofs, the "fundamental theorem of algebraic equations." It is without doubt one of the milestones in the history of mathematics. As Reinhold Remmert writes in his pertinent survey: "It was the possibility of proving this theorem in the complex domain that, more than anything else, paved the way for a general recognition of complex numbers."

Some of the greatest names have contributed to the subject, from Gauss and Cauchy to Liouville and Laplace. An article of Netto and Le Vasseur lists nearly a hundred proofs. The proof that we present is one of the most elegant and certainly the shortest. It follows an argument of d'Alembert and Argand and uses only some elementary properties of polynomials and complex numbers. We are indebted to France Dacar for a polished version of the proof. Essentially the same argument appears also in the papers of Redheffer [4] and Wolfenstein [6], and doubtlessly in some others.


Jean Le Rond d'Alembert

We need three facts that one learns in a first-year calculus course.

(A) Polynomial functions are continuous.

(B) Any complex number of absolute value 1 has an m-th root for any $m \geq 1$.

(C) Cauchy's minimum principle: A continuous real-valued function f on a compact set S assumes a minimum in S.

Now let $p(z) = \sum_{k=0}^{n} c_k z^k$ be a complex polynomial of degree $n \geq 1$. As the first and decisive step we prove what is variously called d'Alembert's lemma or Argand's inequality.

Lemma. *If $p(a) \neq 0$, then every disc D around a contains an interior point b with $|p(b)| < |p(a)|$.*

■ **Proof.** Suppose the disc D has radius R. Thus the points in the interior of D are of the form $a + w$ with $|w| < R$. First we will show that by a simple algebraic manipulation we may write $p(a + w)$ as

$$p(a+w) \quad = \quad p(a) + cw^m(1 + r(w)), \tag{1}$$

where c is a nonzero complex number, $1 \leq m \leq n$, and $r(w)$ is a polynomial of degree $n - m$ with $r(0) = 0$.

Indeed, we have

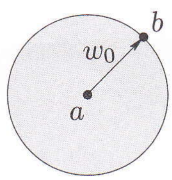
$$\begin{aligned} p(a+w) &= \sum_{k=0}^{n} c_k (a+w)^k \\ &= \sum_{k=0}^{n} c_k \sum_{i=0}^{k} \binom{k}{i} a^{k-i} w^i \;=\; \sum_{i=0}^{n} \Big(\sum_{k=i}^{n} \binom{k}{i} c_k a^{k-i} \Big) w^i \\ &= p(a) + \sum_{i=1}^{n} \Big(\sum_{k=i}^{n} \binom{k}{i} c_k a^{k-i} \Big) w^i \;=\; p(a) + \sum_{i=1}^{n} d_i w^i . \end{aligned}$$

Now let $m \geq 1$ be the minimal index i for which d_i is different from zero, set $c = d_m$, and factor cw^m out to obtain

$$p(a+w) \;=\; p(a) + cw^m(1 + r(w)).$$

Next we want to bound $|cw^m|$ and $|r(w)|$ from above. If $|w|$ is smaller than $\rho_1 := \sqrt[m]{|p(a)/c|}$, then $|cw^m| < |p(a)|$. Further, since $r(w)$ is continuous and $r(0) = 0$, we have $|r(w)| < 1$ for $|w| < \rho_2$. Hence for $|w|$ smaller than $\rho := \min(\rho_1, \rho_2)$ we have

$$|cw^m| \;<\; |p(a)| \qquad \text{and} \qquad |r(w)| \;<\; 1. \tag{2}$$



We come to our second ingredient, m-th roots of unity. Let ζ be an m-th root of $-\frac{p(a)/c}{|p(a)/c|}$, which is a complex number of absolute value 1. Let ε be a real number with $0 < \varepsilon < \min(\rho, R)$. Setting $w_0 = \varepsilon\zeta$ we claim that $b = a + w_0$ is a desired point in the disc D with $|p(b)| < |p(a)|$. First of all, b is in D, since $|w_0| = \varepsilon < R$, and further by (1),

$$|p(b)| = |p(a + w_0)| = |p(a) + cw_0^m(1 + r(w_0))|. \tag{3}$$

Now we define a factor δ by

$$c\,w_0^m \;=\; c\,\varepsilon^m \zeta^m \;=\; -\frac{\varepsilon^m}{|p(a)/c|}\,p(a) \;=\; -\delta\,p(a),$$

where by (2) our δ satisfies

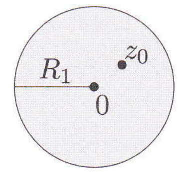
$$0 \;<\; \delta \;=\; \varepsilon^m \frac{|c|}{|p(a)|} \;<\; 1.$$

With the triangle inequality we get for the right-hand term in (3)

$$\begin{aligned} |p(a) + cw_0^m(1 + r(w_0))| &= |p(a) - \delta p(a)(1 + r(w_0))| \\ &= |(1-\delta)p(a) - \delta p(a) r(w_0)| \\ &\leq (1-\delta)|p(a)| \;+\; \delta |p(a)||r(w_0)| \\ &< (1-\delta)|p(a)| \;+\; \delta|p(a)| \;=\; |p(a)|, \end{aligned}$$

and we are done. □

The rest is easy. Clearly, $p(z)z^{-n}$ approaches the leading coefficient c_n of $p(z)$ as $|z|$ goes to infinity. Hence $|p(z)|$ goes to infinity as well with $|z| \to \infty$. Consequently, there exists $R_1 > 0$ such that $|p(z)| > |p(0)|$ for all points z on the circle $\{z : |z| = R_1\}$. Furthermore, our third fact (C) tells us that in the compact set $D_1 = \{z : |z| \leq R_1\}$ the continuous real-valued function $|p(z)|$ attains the minimum value at some point z_0. Because of $|p(z)| > |p(0)|$ for z on the boundary of D_1, z_0 must lie in the interior. But by d'Alembert's lemma this minimum value $|p(z_0)|$ must be 0 — and this is the whole proof.



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[3] E. NETTO AND R. LE VAVASSEUR: *Les fonctions rationelles,* Enc. Sciences Math. Pures Appl. I **2** (1907), 1-232.

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"What's up this time?"

"Well, I'm shlepping 100 proofs for the Fundamental Theorem of Algebra"

"Proofs from the Book: one for the Fundamental Theorem, one for Quadratic Reciprocity!"