Math 3325 **Test 2** **Spring 2013**

You may use only paper and pencil. If you use scratch paper, first write your name on top of each page and then turn it in together with this test.

The test has 5 problems on 5 pages.

Problem 1 (20 points) 1. Give the definition for the union of two sets.

$A \cup B = \{x \mid x \in A \ \vee \ x \in B\}$.

2. Let A, B and C be sets. Prove or disprove:

If $A \subseteq B \cup C$, then $A \subseteq B$ or $A \subseteq C$.

The statement is false. Counterexample: $A = \{1, 2\}$, $B = \{1\}$, $C = \{2\}$.

Problem 2 (20 points) 1. Let A be a set. Give the definition for the power set $\mathcal{P}(A)$ of A.

The power set of a set A is the set of all subsets of A.

2. Let A and B be two sets. Show:

If $A \cap B = \emptyset$, then $\mathcal{P}(A) \cap \mathcal{P}(B) = \{\emptyset\}$.

Note first that $\emptyset$ is an element of any power set.

Now let $C \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Then $C \in \mathcal{P}(A)$ and $C \in \mathcal{P}(B)$. In other words, $C \subseteq A$ and $C \subseteq B$. Thus $C \subseteq A \cap B$. By our hypothesis we conclude that $C = \emptyset$.

Remark: It is indeed true that $C \subseteq A$ and $C \subseteq B$ implies $C \subseteq A \cap B$. To see this, let $x \in C$. Then $x \in A$ and $x \in B$, and thus $x \in A \cap B$.

3. State the converse of 2.

If $\mathcal{P}(A) \cap \mathcal{P}(B) = \{\emptyset\}$, then $A \cap B = \emptyset$.

4. Prove the converse of 2.

Suppose to the contrary that $A \cap B \neq \emptyset$, say $x \in A \cap B$. Then $\{x\} \subseteq A$ and $\{x\} \subseteq B$. Thus $\{x\} \in \mathcal{P}(A)$ and $\{x\} \in \mathcal{P}(B)$, so $\{x\} \in \mathcal{P}(A) \cap \mathcal{P}(B)$. We conclude that $\mathcal{P}(A) \cap \mathcal{P}(B) \neq \{\emptyset\}$.

Problem 3 (20 points) 1. Let I be an index set, and let $\{A_i \mid i \in I\}$ be a collection of sets. Give the definition for $\bigcap\limits_{i \in I} A_i$.

$$\bigcap_{i \in I} A_i = \{x \mid x \in A_i \text{ for all } i \in I\}.$$

2. Let B be another set. Show:

$$\left(\bigcap_{i\in I} A_i\right) \cup B = \bigcap_{i\in I}(A_i \cup B).$$

1. Let $x \in \left(\bigcap_{i\in I} A_i\right) \cup B$.

   Case 1: $x \in B$. Then $x \in A_i \cup B$ for all $i \in I$; consequently $x \in \bigcap_{i\in I}(A_i \cup B)$.

   Case 2: $x \in \left(\bigcap_{i\in I} A_i\right)$. Then $x \in A_i$ for all $i \in I$, and therefore $x \in A_i \cup B$ for all $i \in I$. Thus we obtain that $x \in \bigcap_{i\in I}(A_i \cup B)$.

2. Now let $x \in \bigcap_{i\in I}(A_i \cup B)$. This means that $x \in A_i \cup B$ for all $i \in I$.

   If $x \notin B$, then $x \in A_i$ for all $i \in I$, and thus $x \in \left(\bigcap_{i\in I} A_i\right) \subseteq \left(\bigcap_{i\in I} A_i\right) \cup B$.

   If, on the other hand, $x \in B$, then we see directly that $x \in \left(\bigcap_{i\in I} A_i\right) \cup B$.

Problem 4 (20 points) For each irrational number $q \in \mathbb{R} \setminus \mathbb{Q}$, let $B_q = \{x \in \mathbb{R} \mid x \neq q\}$.

1. Find $\bigcup_{q\in\mathbb{R}\setminus\mathbb{Q}} B_q$. Confirm your conjecture by a proof.

   $\bigcup_{q\in\mathbb{R}\setminus\mathbb{Q}} B_q = \mathbb{R}$.

   Indeed $B_{\sqrt{2}} = \mathbb{R} \setminus \{\sqrt{2}\}$ and $\sqrt{2} \in B_{\sqrt{3}}$. Thus $\mathbb{R} = B_{\sqrt{2}} \cup B_{\sqrt{3}} \subseteq \bigcup_{q\in\mathbb{R}\setminus\mathbb{Q}} B_q$.

2. Find $\bigcap_{q\in\mathbb{R}\setminus\mathbb{Q}} B_q$. Confirm your conjecture by a proof.

   $\bigcap_{q\in\mathbb{R}\setminus\mathbb{Q}} B_q = \mathbb{Q}$.

   If $x \in \mathbb{Q}$, then $x \neq q$ for all $q \in \mathbb{R} \setminus \mathbb{Q}$, and thus $x \in B_q$ for all $q \in \mathbb{R} \setminus \mathbb{Q}$. If, on the other hand, $x \in \mathbb{R} \setminus \mathbb{Q}$, then $x \notin B_x$, and thus $x \notin \bigcap_{q\in\mathbb{R}\setminus\mathbb{Q}} B_q$.

Problem 5 (20 points) Use the *Principle of Mathematical Induction* to show the following holds for all $n \in \mathbb{N}$:

$$\prod_{i=1}^{n}\left(1 - \frac{1}{i+1}\right) = \frac{1}{n+1}$$

For $n = 1$, both sides of the equation evaluate to $\frac{1}{2}$. Now let $n \in \mathbb{N}$ and assume that

$$\prod_{i=1}^{n} \left(1 - \frac{1}{i+1}\right) = \frac{1}{n+1}. \tag{1}$$

Then we obtain for $n + 1$:

$$\begin{aligned}
\prod_{i=1}^{n+1} \left(1 - \frac{1}{i+1}\right) &= \left[\prod_{i=1}^{n} \left(1 - \frac{1}{i+1}\right)\right] \cdot \left(1 - \frac{1}{n+2}\right) \\
&= \frac{1}{n+1} \cdot \left(1 - \frac{1}{n+2}\right) && \text{by (1)} \\
&= \frac{1}{n+1} \cdot \frac{n+1}{n+2} \\
&= \frac{1}{n+2}.
\end{aligned}$$