$$\exists N \geq 0 \quad \forall n \geq N \qquad \sqrt[n]{|a_n|} \leq q.$$

This implies $|a_n| \leq q^n$ for $n \geq N$, and a comparison with the geometric series yields the absolute convergence of $\sum_{i=0}^{\infty} a_i$. If $\limsup_{n\to\infty} \sqrt[n]{|a_n|} > 1$, then the condition (2.3) is not satisfied and the series cannot converge. □

Double Series

Consider a two-dimensional array of real numbers

$$(2.13)\qquad \begin{array}{ccccccccccc}
a_{00} & + & a_{01} & + & a_{02} & + & a_{03} & +\ldots & = & s_0 \\
+ & & + & & + & & + & & & + \\
a_{10} & + & a_{11} & + & a_{12} & + & a_{13} & +\ldots & = & s_1 \\
+ & & + & & + & & + & & & + \\
a_{20} & + & a_{21} & + & a_{22} & + & a_{23} & +\ldots & = & s_2 \\
+ & & + & & + & & + & & & + \\
a_{30} & + & a_{31} & + & a_{32} & + & a_{33} & +\ldots & = & s_3 \\
+ & & + & & + & & + & & & + \\
\vdots & & \vdots & & \vdots & & \vdots & & & \vdots \\
= & & = & & = & & = & & & = \\
v_0 & + & v_1 & + & v_2 & + & v_3 & +\ldots & = & ???
\end{array}$$

and suppose we want to sum up all of them. There are many natural ways of doing this. One can either add up the elements of the ith row, denote the result by s_i, and then compute $\sum_{i=0}^{\infty} s_i$; or one can add up the elements of the jth column, denote the result by v_j, and then compute $\sum_{j=0}^{\infty} v_j$. It is also possible to write all elements in a linear arrangement. For example, we can start with a_{00}, then add the elements a_{ij} for which $i+j=1$, then those with $i+j=2$, and so on. This gives

$$(2.14)\qquad a_{00} + (a_{10} + a_{01}) + (a_{20} + a_{11} + a_{02}) + (a_{30} + \ldots) + \ldots .$$

Here, we denote the pairs $(0,0)$, $(1,0)$, $(0,1)$, $(2,0)$, ... by $\sigma(0)$, $\sigma(1)$, $\sigma(2)$, $\sigma(3)$, ..., so that σ is a map $\sigma : \mathbb{N}_0 \to \mathbb{N}_0 \times \mathbb{N}_0$, where $\mathbb{N}_0 \times \mathbb{N}_0 = \{(i,j) \mid i \in \mathbb{N}_0, j \in \mathbb{N}_0\}$ is the so-called Cartesian product of $\mathbb{N}_0$ with $\mathbb{N}_0$. So, we define in general,

(2.12) Definition. *A series $\sum_{k=0}^{\infty} b_k$ is called a linear arrangement of the double series (2.13) if there exists a bijective mapping $\sigma : \mathbb{N}_0 \to \mathbb{N}_0 \times \mathbb{N}_0$ such that $b_k = a_{\sigma(k)}$.*

The question now is: do the different possibilities of summation lead to the same value? Do we have

$$(2.15)\qquad s_0 + s_1 + \ldots = \sum_{i=0}^{\infty}\Big(\sum_{j=0}^{\infty} a_{ij}\Big) = \sum_{j=0}^{\infty}\Big(\sum_{i=0}^{\infty} a_{ij}\Big) = v_0 + v_1 + \ldots ,$$

and do linear arrangements converge to the same value?

The counterexample of Fig. 2.4a shows that this is not true without some additional assumptions.

$$\begin{array}{ccccccc}
1 & - 1 & + 0 & + 0 & + \ldots = & 0 \\
+ & + & + & + & & + \\
0 & + 1 & - 1 & + 0 & + \ldots = & 0 \\
+ & + & + & + & & + \\
0 & + 0 & + 1 & - 1 & + \ldots = & 0 \\
+ & + & + & + & & + \\
0 & + 0 & + 0 & + 1 & - \ldots = & 0 \\
+ & + & + & + & & + \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
= & = & = & = & & = \\
1 & + 0 & + 0 & + 0 & + \ldots = & 1 \neq 0
\end{array}$$

FIGURE 2.4a. Counterexample

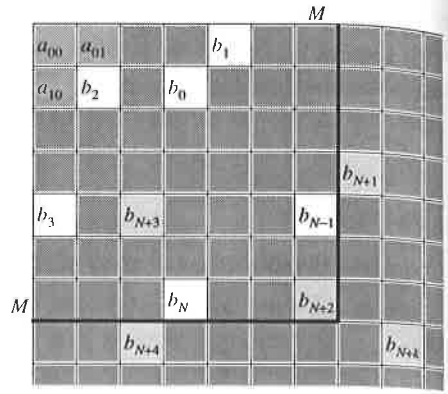

FIGURE 2.4b. Double series

(2.13) Theorem (Cauchy 1821, "Note VII"). *Suppose for the double series (2.13) that*

$$\exists B \geq 0 \quad \forall m \geq 0 \quad \sum_{i=0}^{m} \sum_{j=0}^{m} |a_{ij}| \leq B. \tag{2.16}$$

Then, all the series in (2.15) are convergent and the identities of (2.15) are satisfied. Furthermore, every linear arrangement of the double series converges to the same value.

Proof. Let $b_0 + b_1 + b_2 + \ldots$ be a linear arrangement of the double series (2.13). The sequence $\{\sum_{i=0}^{n} |b_i|\}$ is monotonically increasing and bounded (by assumption (2.16)) so that $\sum_{i=0}^{\infty} |b_i|$, and hence also $\sum_{i=0}^{\infty} b_i$, converge. Analogously, we can establish the convergence of $s_i = \sum_{j=0}^{\infty} a_{ij}$ and $v_j = \sum_{i=0}^{\infty} a_{ij}$.

Inspired by the proof of Theorem 2.9, we apply Cauchy's criterion to the series $\sum_{i=0}^{\infty} |b_i|$ and have

$$\forall \varepsilon > 0 \quad \exists N \geq 0 \quad \forall n \geq N \quad \forall k \geq 1 \quad |b_{n+1}| + |b_{n+2}| + \ldots + |b_{n+k}| < \varepsilon.$$

For a given $\varepsilon > 0$ and the corresponding $N \geq 0$ we choose an integer M in such a way that all elements $b_0, b_1, \ldots, b_N$ are present in the box $0 \leq i \leq M$, $0 \leq j \leq M$ (see Fig. 2.4b). With this choice, $b_0, b_1, \ldots, b_N$ appear in the sum $\sum_{i=0}^{l} b_i$ (for $l \geq N$) as well as in $\sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij}$ (for $m \geq M$ and $n \geq M$). Hence, we have for $l \geq N$, $m \geq M$, $n \geq M$,

$$\left| \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} - \sum_{i=0}^{l} b_i \right| \leq |b_{N+1}| + \ldots + |b_{N+k}| < \varepsilon, \tag{2.17}$$

with a sufficiently large k. We set $s = \sum_{i=0}^{\infty} b_i$ and take the limits $l \to \infty$ and $n \to \infty$ in (2.17). Then, we exchange the finite summations $\sum_{i=0}^{m} \sum_{j=0}^{n} \leftrightarrow$

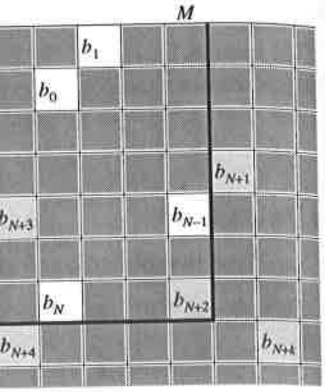

FIGURE 2.4b. Double series

ose for the double series (2.13)

$|a_{ij}| \le B.$

he identities of (2.15) are satis-
e double series converges to the

t of the double series (2.13). The
g and bounded (by assumption
converge. Analogously, we can
$v_j = \sum_{i=0}^{\infty} a_{ij}$.
apply Cauchy's criterion to the

$| + |b_{n+2}| + \ldots + |b_{n+k}| < \varepsilon.$

0 we choose an integer M in
resent in the box $0 \le i \le M$,
$b_0, b_1, \ldots, b_N$ appear in the sum
a_{ij} (for $m \ge M$ and $n \ge M$).

$\ldots + |b_{N+k}| < \varepsilon,$

and take the limits $l \to \infty$ and
ite summations $\sum_{i=0}^{m} \sum_{j=0}^{n} \leftrightarrow$

$\sum_{j=0}^{n} \sum_{i=0}^{m}$ and take the limits $l \to \infty$ and $m \to \infty$. This yields, by Theorem 1.6,

$$\left|\sum_{i=0}^{m} s_i - s\right| \le \varepsilon \quad \text{and} \quad \left|\sum_{j=0}^{n} v_j - s\right| \le \varepsilon.$$

Hence $\sum_{i=0}^{\infty} s_i$ and $\sum_{j=0}^{\infty} v_j$ both converge to the same limit s. □

The Cauchy Product of Two Series

If we want to compute the product of two infinite series $\sum_{i=0}^{\infty} a_i$ and $\sum_{j=0}^{\infty} b_j$, we have to add all elements of the two-dimensional array

$$\begin{array}{ccccc} a_0b_0 & a_0b_1 & a_0b_2 & a_0b_3 & \ldots \\ a_1b_0 & a_1b_1 & a_1b_2 & a_1b_3 & \ldots \\ a_2b_0 & a_2b_1 & a_2b_2 & a_2b_3 & \ldots \\ a_3b_0 & a_3b_1 & a_3b_2 & a_3b_3 & \ldots . \\ \vdots & \vdots & \vdots & \vdots & \end{array} \tag{2.18}$$

If we arrange the elements as indicated in Eq. (2.14), we obtain the so-called Cauchy product of the two series.

(2.14) Definition. *The Cauchy product of the series* $\sum_{i=0}^{\infty} a_i$ *and* $\sum_{j=0}^{\infty} b_j$ *is defined by*

$$\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} a_{n-j} \cdot b_j\right) = a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \ldots .$$

The question is whether the Cauchy product is a convergent series and whether it really represents the product of the two series $\sum_{i\ge 0} a_i$ and $\sum_{j\ge 0} b_j$.

(2.15) Counterexample (Cauchy 1821). The series

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} - \ldots$$

converges by Leibniz's criterion. We consider the Cauchy product of this series with itself. Since

$$\left|\sum_{j=0}^{n} a_{n-j} \cdot b_j\right| = \sum_{j=0}^{n} \frac{1}{\sqrt{n+1-j} \cdot \sqrt{j+1}} \ge \frac{2n+2}{n+2}$$

(the inequality is a consequence of $(n+1-x)(x+1) \le (1+n/2)^2$ for $0 \le x \le n$), the necessary condition (2.3) for the convergence of the Cauchy product is not satisfied (see Fig. 2.5). This example illustrates the fact that the Cauchy product of two convergent series need not converge.

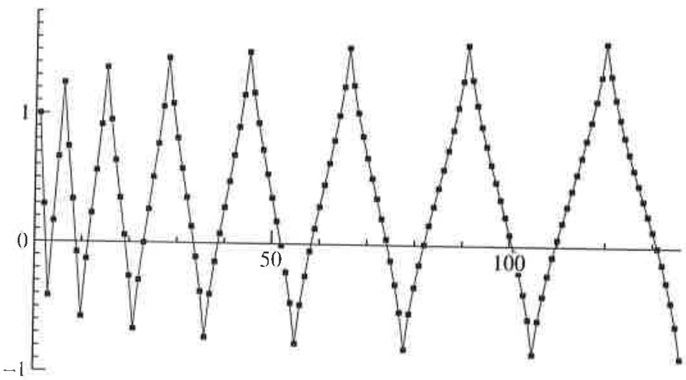

FIGURE 2.5. Divergence of the Cauchy product of Counterexample 2.15

(2.16) Theorem (Cauchy 1821). *If the two series $\sum_{i=0}^{\infty} a_i$ and $\sum_{j=0}^{\infty} b_j$ are absolutely convergent, then its Cauchy product converges and we have*

$$\left(\sum_{i=0}^{\infty} a_i\right)\cdot\left(\sum_{j=0}^{\infty} b_j\right) = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} a_{n-j}\cdot b_j\right). \tag{2.19}$$

Proof. By hypothesis, we have $\sum_{i=0}^{\infty}|a_i| \le B_1$ and $\sum_{j=0}^{\infty}|b_j| \le B_2$. Therefore, we have for the two-dimensional array (2.18) that for all $m \ge 0$

$$\sum_{i=0}^{m}\sum_{j=0}^{m}|a_i||b_j| \le B_1 B_2,$$

and Theorem 2.13 can be applied. The sum of the ith row gives $s_i = a_i \cdot \sum_{j=0}^{\infty} b_j$ and $\sum_{i=0}^{\infty} s_i = (\sum_{i=0}^{\infty} a_i)(\sum_{j=0}^{\infty} b_j)$. By Theorem 2.13, the Cauchy product, which is a linear arrangement of (2.18), also converges to this value. □

Examples. For $|q| < 1$ consider the two series

$$1 + q + q^2 + q^3 + \ldots = \frac{1}{1-q} \qquad\text{and}\qquad 1 - q + q^2 - q^3 + \ldots = \frac{1}{1+q}.$$

Their Cauchy product is

$$1 + q^2 + q^4 + q^6 + \ldots = \frac{1}{1-q^2},$$

which, indeed, is the product of $(1-q)^{-1}$ and $(1+q)^{-1}$.

The Cauchy product of the absolutely convergent series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \qquad\text{and}\qquad e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots$$

gives the series for e^{x+y} (use the binomial identity of Theorem I.2.1).

Remark. The statement of Theorem 2.16 remains true if only one of the two series is absolutely convergent and the second is convergent (F. Mertens 1875, see Exercise 2.3).

Under the assumption that the series $\sum_i a_i$, $\sum_j b_j$ and also their Cauchy product (Definition 2.14) converge, the identity (2.19) holds (Abel 1826, see Exercise 7.9).

Exchange of Infinite Series and Limits

At several places in Chap. I, we were confronted with the problem of exchanging an infinite series with a limit (for example, for the derivation of the series for e^x in Sect. I.2 and of those for $\sin x$ and $\cos x$ in Sect. I.4). We considered series $d_n = \sum_{j=0}^{\infty} s_{nj}$ depending on an integer parameter n, and used the fact that $\lim_{n\to\infty} d_n = \sum_{j=0}^{\infty} \lim_{n\to\infty} s_{nj}$. Already in Sect. I.2 (after Eq. (I.2.17)), it was observed that this is not always true and that some caution is necessary. The following theorem states sufficient conditions for the validity of such an exchange.

(2.17) Theorem. *Suppose that the elements of the sequence* $\{s_{0j}, s_{1j}, s_{2j}, \ldots\}$ *all have the same sign and that* $|s_{n+1,j}| \geq |s_{nj}|$ *for all* n *and* j. *If there exists a bound* B *such that* $\sum_{j=0}^{n} |s_{nj}| \leq B$ *for all* $n \geq 0$, *then*

$$\lim_{n\to\infty} \sum_{j=0}^{\infty} s_{nj} = \sum_{j=0}^{\infty} \lim_{n\to\infty} s_{nj}. \tag{2.20}$$

Proof. The idea is to reformulate the hypotheses in such a way that Theorem 2.13 is directly applicable. At the beginning of this section, we saw that every series can be converted to an infinite sequence by considering the partial sums (2.2). Conversely, if the partial sums $s_0, s_1, s_2, \ldots$ are given, we can uniquely define elements a_i such that $\sum_{i=0}^{n} a_i = s_n$. We just have to set $a_0 = s_0$ and $a_i = s_i - s_{i-1}$ for $i \geq 1$.

Applying this idea to the sequence $\{s_{0j}, s_{1j}, s_{2j}, \ldots\}$, we define

$$a_{0j} := s_{0j}, \quad a_{ij} := s_{ij} - s_{i-1,j}, \quad \text{so that} \quad \sum_{i=0}^{n} a_{ij} = s_{nj}.$$

Replacing s_{nj} by this expression, (2.20) becomes

$$\lim_{n\to\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{n} a_{ij} = \sum_{j=0}^{\infty} \lim_{n\to\infty} \sum_{i=0}^{n} a_{ij}. \tag{2.21}$$

Exchanging the summations in the expression on the left side of (2.21) (this is permitted by Theorem 1.5), we see that (2.21) is equivalent to (2.15). Therefore, we only have to verify condition (2.16). The assumptions on $\{s_{0j}, s_{1j}, \ldots\}$ imply that the elements $a_{0j}, a_{1j}, \ldots$ all have the same sign. Hence, we have