Since

$$s_n = 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \cdots + \frac{1}{1\cdot 2\cdots n}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 3,$$

the series converges, and the definition makes sense. In fact, the series converges very rapidly and allows us to compute e with great accuracy.

It is of interest to note that e can also be defined by means of another limit process; the proof provides a good illustration of operations with limits:

3.31 Theorem $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e.$

Proof Let

$$s_n = \sum_{k=0}^{n} \frac{1}{k!}, \qquad t_n = \left(1 + \frac{1}{n}\right)^n.$$

By the binomial theorem,

$$t_n = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots$$

$$+ \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right).$$

Hence $t_n \le s_n$, so that

(14) $$\limsup_{n\to\infty} t_n \le e,$$

by Theorem 3.19. Next, if $n \ge m$,

$$t_n \ge 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{m!}\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{m-1}{n}\right).$$

Let $n \to \infty$, keeping m fixed. We get

$$\liminf_{n\to\infty} t_n \ge 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!},$$

so that

$$s_m \le \liminf_{n\to\infty} t_n.$$

Letting $m \to \infty$, we finally get

(15) $$e \le \liminf_{n\to\infty} t_n.$$

The theorem follows from (14) and (15).

The rapidity with which the series $\sum \frac{1}{n!}$ converges can be estimated as follows: If s_n has the same meaning as above, we have

$$e - s_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots$$

$$< \frac{1}{(n+1)!}\left\{1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right\} = \frac{1}{n!n}$$

so that

(16) $$0 < e - s_n < \frac{1}{n!n}.$$

Thus s_{10}, for instance, approximates e with an error less than 10^{-7}. The inequality (16) is of theoretical interest as well, since it enables us to prove the irrationality of e very easily.

3.32 Theorem *e is irrational.*

Proof Suppose e is rational. Then $e = p/q$, where p and q are positive integers. By (16),

(17) $$0 < q!(e - s_q) < \frac{1}{q}.$$

By our assumption, $q!e$ is an integer. Since

$$q!s_q = q!\left(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{q!}\right)$$

is an integer, we see that $q!(e - s_q)$ is an integer.

Since $q \ge 1$, (17) implies the existence of an integer between 0 and 1. We have thus reached a contradiction.

Actually, e is not even an algebraic number. For a simple proof of this, see page 25 of Niven's book, or page 176 of Herstein's, cited in the Bibliography.

THE ROOT AND RATIO TESTS

3.33 Theorem (Root Test) *Given Σa_n, put $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$.*

Then

(*a*) *if $\alpha < 1$, Σa_n converges;*
(*b*) *if $\alpha > 1$, Σa_n diverges;*
(*c*) *if $\alpha = 1$, the test gives no information.*

Proof If $\alpha < 1$, we can choose β so that $\alpha < \beta < 1$, and an integer N such that

$$\sqrt[n]{|a_n|} < \beta$$

for $n \geq N$ [by Theorem 3.17(*b*)]. That is, $n \geq N$ implies

$$|a_n| < \beta^n.$$

Since $0 < \beta < 1$, $\Sigma\beta^n$ converges. Convergence of Σa_n follows now from the comparison test.

If $\alpha > 1$, then, again by Theorem 3.17, there is a sequence $\{n_k\}$ such that

$$\sqrt[n_k]{|a_{n_k}|} \to \alpha.$$

Hence $|a_n| > 1$ for infinitely many values of n, so that the condition $a_n \to 0$, necessary for convergence of Σa_n, does not hold (Theorem 3.23).

To prove (*c*), we consider the series

$$\sum \frac{1}{n}, \sum \frac{1}{n^2}.$$

For each of these series $\alpha = 1$, but the first diverges, the second converges.

3.34 Theorem (Ratio Test) *The series* Σa_n

(*a*) *converges if* $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$,

(*b*) *diverges if* $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ *for* $n \geq n_0$, *where* n_0 *is some fixed integer.*

Proof If condition (*a*) holds, we can find $\beta < 1$, and an integer N, such that

$$\left|\frac{a_{n+1}}{a_n}\right| < \beta$$

for $n \geq N$. In particular,

$$|a_{N+1}| < \beta|a_N|,$$
$$|a_{N+2}| < \beta|a_{N+1}| < \beta^2|a_N|,$$
$$\dots\dots\dots\dots\dots\dots$$
$$|a_{N+p}| < \beta^p|a_N|.$$

That is,

$$|a_n| < |a_N|\beta^{-N} \cdot \beta^n$$

for $n \geq N$, and (*a*) follows from the comparison test, since $\Sigma\beta^n$ converges.

If $|a_{n+1}| \geq |a_n|$ for $n \geq n_0$, it is easily seen that the condition $a_n \to 0$ does not hold, and (*b*) follows.

Note: The knowledge that $\lim a_{n+1}/a_n = 1$ implies nothing about the convergence of Σa_n. The series $\Sigma 1/n$ and $\Sigma 1/n^2$ demonstrate this.

3.35 Examples

(*a*) Consider the series

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots,$$

for which

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0,$$

$$\liminf_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[2n]{\frac{1}{3^n}} = \frac{1}{\sqrt{3}},$$

$$\limsup_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[2n]{\frac{1}{2^n}} = \frac{1}{\sqrt{2}},$$

$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{3}{2}\right)^n = +\infty.$$

The root test indicates convergence; the ratio test does not apply.

(*b*) The same is true for the series

$$\frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16} + \frac{1}{128} + \frac{1}{64} + \cdots,$$

where

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{8},$$

$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = 2,$$

but

$$\lim \sqrt[n]{a_n} = \tfrac{1}{2}.$$

3.36 Remarks The ratio test is frequently easier to apply than the root test, since it is usually easier to compute ratios than nth roots. However, the root test has wider scope. More precisely: Whenever the ratio test shows convergence, the root test does too; whenever the root test is inconclusive, the ratio test is too. This is a consequence of Theorem 3.37, and is illustrated by the above examples.

Neither of the two tests is subtle with regard to divergence. Both deduce divergence from the fact that a_n does not tend to zero as $n \to \infty$.

3.37 Theorem *For any sequence $\{c_n\}$ of positive numbers,*

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \le \liminf_{n\to\infty} \sqrt[n]{c_n},$$

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \le \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

Proof We shall prove the second inequality; the proof of the first is quite similar. Put

$$\alpha = \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

If $\alpha = +\infty$, there is nothing to prove. If α is finite, choose $\beta > \alpha$. There is an integer N such that

$$\frac{c_{n+1}}{c_n} \le \beta$$

for $n \ge N$. In particular, for any $p > 0$,

$$c_{N+k+1} \le \beta c_{N+k} \qquad (k = 0, 1, \ldots, p-1).$$

Multiplying these inequalities, we obtain

$$c_{N+p} \le \beta^p c_N,$$

or

$$c_n \le c_N \beta^{-N} \cdot \beta^n \qquad (n \ge N).$$

Hence

$$\sqrt[n]{c_n} \le \sqrt[n]{c_N \beta^{-N}} \cdot \beta,$$

so that

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \le \beta, \tag{18}$$

by Theorem 3.20(b). Since (18) is true for every $\beta > \alpha$, we have

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \le \alpha.$$

POWER SERIES

3.38 Definition Given a sequence $\{c_n\}$ of complex numbers, the series

$$\sum_{n=0}^{\infty} c_n z^n \tag{19}$$

is called a *power series*. The numbers c_n are called the *coefficients* of the series; z is a complex number.

In general, the series will converge or diverge, depending on the choice of z. More specifically, with every power series there is associated a circle, the circle of convergence, such that (19) converges if z is in the interior of the circle and diverges if z is in the exterior (to cover all cases, we have to consider the plane as the interior of a circle of infinite radius, and a point as a circle of radius zero). The behavior on the circle of convergence is much more varied and cannot be described so simply.

3.39 Theorem *Given the power series $\Sigma c_n z^n$, put*

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}, \qquad R = \frac{1}{\alpha}.$$

(If $\alpha = 0$, $R = +\infty$; if $\alpha = +\infty$, $R = 0$.) Then $\Sigma c_n z^n$ converges if $|z| < R$, and diverges if $|z| > R$.

Proof Put $a_n = c_n z^n$, and apply the root test:

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = |z| \limsup_{n\to\infty} \sqrt[n]{|c_n|} = \frac{|z|}{R}.$$

Note: R is called the radius of convergence of $\Sigma c_n z^n$.

3.40 Examples

(a) The series $\Sigma n^n z^n$ has $R = 0$.

(b) The series $\sum \frac{z^n}{n!}$ has $R = +\infty$. (In this case the ratio test is easier to apply than the root test.)