HOW TO WRITE MATHEMATICS

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The purpose of this introductory guide is to help you write mathematical arguments. Good mathematical writing takes practice; it is also necessary to know some basic rules. Perhaps the most important feature of good mathematical writing is the revision process: writing and rewriting. This guide discusses, with examples, the principles of mathematical writing.

Contents

1 What is the goal of mathematical writing?

As a mathematics major, you need to know many things in order to get started in the world of mathematics. These things include mathematical concepts, definitions, theorems, and proofs. Equally important is the knowledge of how to *write* your ideas, solutions, and proofs. Some students are surprised by this. They ask, "Why do I need to learn to write if I am a mathematics major?" The answer is that writing is important in just about all fields, and certainly in mathematics. You need to learn to write so that others can follow your work.

The goal of mathematical writing is clear communication of mathematical ideas. Mathematical writing is accurate, precise, and concise. In general, writing is a skill that can and should be worked on, and it is a skill that can be improved with practice. Writing about a subject goes hand-in-hand with learning about the subject. Mathematics is a challenging field in which these general principles of writing are of critical importance.

2 General principles of mathematical writing

Here we give an overview of the principles of mathematical writing. We will cover these principles in more detail in the later sections. Good resources on mathematical writing are [1] and [3].

Remember to practice the three most important principles of good writing:

- Say something worthwhile.
- Proofread.
- Revise.

Mathematical writing has some further requirements. Here are some essential rules of mathematical writing:

- Write complete sentences. Writing complete sentences helps you to organize your thoughts and convey what you want to convey in the clearest way.
- Write accurately and precisely. Do not write opinions, meaningless examples, or extraneous expressions. Avoid using the words "you" and "I" in math proofs.
- Avoid overly wordy and overly symbolic writing. Use a balance of words and symbols.
- In long solutions or proofs, tell the reader in advance what you are trying to accomplish.
- Pay attention to all aspects of your writing: punctuation, spelling, mathematical content, readability, etc.
- Write and rewrite.

3 Writing mathematical sentences

The basic unit of mathematical writing is the sentence. A good mathematical argument consists of sentences, arranged in paragraphs, that prove a theorem or give an example or illustrate how to solve a problem. Whatever the specific purpose of the mathematics, the goal is to create readable, understandable sentences. Remember that someone (usually other than yourself) must read your work. Try to put yourself in your reader's shoes and make that person's job as easy as possible.

Mathematical sentences are grammatical sentences, with the mathematics inserted in a logical way.

EXAMPLE: Let $S = \{1, 2, 3, 4\}$. There are 16 subsets of S, including S itself and the empty set $\emptyset$.

Notice that the above example contains two complete sentences. Each sentence is a blend of words and symbols.

4 Avoiding common errors

Here are some examples of wrong and right mathematical writing:

1. Begin sentences with a capital letter and end them with a period.

   WRONG: let $x = 5$, then $x + 10 = 15$

   RIGHT: Let $x = 5$. Then $x + 10 = 15$.

2. Start sentences with words, not symbols.

   WRONG: x is a positive real number, so x has a real square root.

   RIGHT: Since x be a positive real number, x has a real square root.

3. Adopt a reader-friendly notation.

   WRONG: Let x be a function from $\mathbf{R}^2$ to $\mathbf{R}^2$. Let F be the set of all such functions.

   RIGHT: Let f be a function from $\mathbf{R}^2$ to $\mathbf{R}^2$. Let S be the set of all such functions.

4. Put things in proper order. Introduce your terms before you use them. Don't use terms that you haven't defined.

   WRONG: Let $x \geq 3$. Then y is a positive real number, where $y = \sqrt{x}$.

   RIGHT: Let $x \geq 3$. Then $y = \sqrt{x}$ is a positive real number.

5. Use a balance of words and symbols.

   WRONG: We know that if we add x and y we get a number greater than or equal to 12.

   RIGHT: We know that $x + y \geq 12$.

6. Write in paragraph style, not tabular style.

   WRONG: Let $x = \pi$.

   Then $x > 3$.

   So x is certainly greater than e.

   RIGHT: Let $x = \pi$. Then $x > 3$, so x is certainly greater than e.

7. Indent long mathematical expressions. Write short mathematical expressions "in-line." Use a balance of displayed equations and in-line equations.

   WRONG: Let $\phi = (a^2+b^2+c^2+d^2)/(e^2+f^2+g^2+h^2) \cdot (a+b+c+d)/(e+f+g+h)$. Then ϕ is a rational function of eight variables.

   RIGHT: Let
   $$\phi = \frac{a^2 + b^2 + c^2 + d^2}{e^2 + f^2 + g^2 + h^2} \cdot \frac{a + b + c + d}{e + f + g + h}.$$
   Then ϕ is a rational function of eight variables.

5 Writing proofs

A mathematical proof is a sequence of sentences that convey a mathematical argument.

Let us create a proof of a simple proposition.

Proposition. Every odd integer can be expressed in exactly one of the forms $4n+1$ or $4n+3$, where n is an integer.

Here is a poor proof:

Proof. An odd integer leaves a remainder of 1 when divided by 2. Now, if we divide the integer by 4, it will leave a remainder of 1 or 3. So the number is of the form $4n+1$ or $4n+3$. □

Why is this bad? There are several reasons. First, the "proof" is heavy-handed: it says that something *will* happen without really explaining why it will happen. Second, it is confused; it talks about "an odd integer" (in the first sentence) and "the integer" (in the second sentence). Third, the n comes out of nowhere at the end; there is no mention of n until its sudden appearance in the last sentence.

Let's try again.

Proof. An odd integer is an integer of the form $2m+1$, where m is an integer which may be even or odd. In the first case, the original integer is of the form $4n+1$, while in the second case it is of the form $4n+3$. □

But this is another wrong turn. What is wrong? We lapsed into an "English-only" mode and forgot to use the notation that we set up. Good mathematical writing is a blend of words and mathematical notation. More precisely, it consists of grammatically correct sentences in which some of the terms are represented by notation.

Notice also how imprecise the above "proof" is. The connection between m and n is unclear.

A good proof explains every step clearly.

Proof. An odd integer can be expressed in the form $2m+1$, where m is an integer. If m is even, then $m=2n$, for some integer n, so that the given integer equals $2(2n)+1=4n+1$. If m is odd, then $m=2n+1$, so that the given integer equals $2(2n+1)+1=4n+3$. □

This is a good proof. Every step is succinctly and correctly explained, and we proved what we set out to prove. Notice that we don't belabor the obvious at the end of proof. It is not necessary to say something like, "Since we started with an arbitrary odd integer, and we showed that it is of the form $4n+1$ or $4n+3$, we are done." Such a statement, while true, is tedious and useless.

There are many fine books about how to write mathematical proofs. See, for example, [4] and [2]. I will close this section with the advice that my advisor, Frank Harary, gave me: "A proof is a proof." By that he meant that a satisfactory proof treats all details fairly and clearly.

6 The revision process

Perhaps the most important element of a mathematical writing endeavor is the revision process. Nobody writes perfect mathematical explanations the first time. We must review our work and rewrite it, striving for greater accuracy, precision, and clarity.

Read your work. That is, after you write something, *read* it and see if it makes sense. Read it aloud. Read it to a friend. Sit quietly in a chair and read it to yourself. You will be surprised to see how often what you wrote doesn't sound right. When this happens you should rewrite your work and read it again.

When reading your work, ask yourself the following questions:

- Does each sentence make sense?

  If not, then fix the sentences that don't make sense.

- Does the logic make common sense?

  If not, then revise the parts that are illogical.

When you have revised your work, read it again until you are satisfied with it. It isn't unusual to make several corrections and revisions.

Since you must revise your work, it is important that you start writing assignments early. I remember a challenging mathematics course that I took, on topology, in which there was a weekly homework assignment consisting of five to ten problems. I found it best to start working on the day that the assignment was given, sketching ideas I had about the problems, trying to find solutions, *and writing* my initial ideas. Later in the week, I would write and rewrite (continuing to try to solve the more difficult problems). By the end of the week—after several revisions—I had a good paper.

If you make good mathematical writing a priority, and follow the principles in this guide, you will improve the quality of your work.

References

[1] N. J. Higham. *Handbook of Writing for the Mathematical Sciences.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1998.

[2] D. E. Knuth, T. Larrabee, and P. M. Roberts. *Mathematical Writing.* Mathematical Association of America, Washington, D.C., revised edition, 1989.

[3] S. G. Krantz. *A Primer of Mathematical Writing: Being a Disquisition on Having Your Ideas Recorded, Typeset, Published, Read, and Appreciated.* American Mathematical Society, Providence, 1997.

[4] D. Smith, M. Eggen, and R. St. Andre. *A Transition to Advanced Mathematics.* Brooks/Cole, Chicago, seventh edition, 2009.