Integrals. Let R be a subset of $\mathbb{R}^2$, and let $\vec{\boldsymbol{r}}(u,v) = (x(u,v), y(u,v), z(u,v)) : R \to S$ parametrize a surface $S \subseteq \mathbb{R}^3$.

Surface area. $\iint_S dA = \iint_R \|\vec{\boldsymbol{r}}_{\boldsymbol{u}} \times \vec{\boldsymbol{r}}_{\boldsymbol{v}}\| du\, dv$.

Surface integral. For a scalar-valued function $f : S \to \mathbb{R}$,

$$\iint_S f\, dA = \iint_R f(\vec{\boldsymbol{r}}(u,v)) \|\vec{\boldsymbol{r}}_{\boldsymbol{u}} \times \vec{\boldsymbol{r}}_{\boldsymbol{v}}\| du\, dv$$

Flux integral. Let $\vec{\boldsymbol{F}} = (F_1, F_2, F_3) : S \to \mathbb{R}^3$ be a vector field, and let $\vec{\boldsymbol{n}}$ be a choice of unit normal for the orientable surface S. Furthermore, let $\alpha = \angle(\vec{\boldsymbol{n}}, \vec{\boldsymbol{i}})$, $\beta = \angle(\vec{\boldsymbol{n}}, \vec{\boldsymbol{j}})$, and $\gamma = \angle(\vec{\boldsymbol{n}}, \vec{\boldsymbol{k}})$. Then

$$\begin{aligned}
\iint_S \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{n}}\, dA &= \iint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma)\, dA \\
&= \iint_R \vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}(u,v)) \cdot (\vec{\boldsymbol{r}}_{\boldsymbol{u}} \times \vec{\boldsymbol{r}}_{\boldsymbol{v}})\, du\, dv \\
&= \pm \iint_R F_1\, dy\, dz \pm \iint_R F_2\, dx\, dz \pm \iint_R F_3\, dx\, dy
\end{aligned}$$

In the last formula, the signs are determined by the signs of $\cos\alpha$, $\cos\beta$, and $\cos\gamma$, respectively.

Integral Theorems in $\mathbb{R}^3$. Let T be a closed bounded solid in $\mathbb{R}^3$ with boundary surface ∂T, oriented such that its unit normal $\vec{\boldsymbol{n}}$ points outward.

Gauss' Divergence Theorem. Let $\vec{\boldsymbol{F}} = (F_1, F_2, F_3) : T \to \mathbb{R}^3$ be a vector field.

$$\iiint_T div\, \vec{\boldsymbol{F}}\, dV = \iint_{\partial T} \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{n}}\; dA.$$

Gradient version of the Divergence Theorem. Let $f : T \to \mathbb{R}$ be a scalar-valued function.

$$\iiint_T \nabla^2 f\, dV = \iint_{\partial T} \frac{\partial f}{\partial \vec{\boldsymbol{n}}}\, dA.$$

Green's Formulas. Assume $f, g : T \to \mathbb{R}$ are scalar-valued functions.

$$\iiint_T (f\, \nabla^2 g + grad\, f \cdot grad\, g)\, dV = \iint_{\partial T} f \frac{\partial g}{\partial \vec{\boldsymbol{n}}}\, dA \tag{1}$$

$$\iiint_T (f\, \nabla^2 g - g\, \nabla^2 f)\, dV = \iint_{\partial T} \left(f \frac{\partial g}{\partial \vec{\boldsymbol{n}}} - g \frac{\partial f}{\partial \vec{\boldsymbol{n}}} \right) dA \tag{2}$$