## 2 Numerical Series

From this section onward the standard results from a first Analysis course are a prerequisite. For this section in particular you can (and will need to) use results about sequences.

Given a sequence $(a_n)_{n\in\mathbb{N}}$, an *infinite series* is a formal expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

The corresponding *sequence of partial sums* $(s_k)_{k\in\mathbb{N}}$ is defined by $s_k = a_1 + a_2 + a_3 + \cdots a_k$.

If the sequence of partial sums converges, with limit $s$, we say that the series $\sum_{n=1}^{\infty} a_n$ converges and we write

$$\sum_{n=1}^{\infty} a_n = s.$$

We will often write $\sum a_n$ instead of $\sum_{n=1}^{\infty} a_n$.

The following are direct consequences of the corresponding fact for sequences:

1. If the series $\sum a_n$ and $\sum b_n$ both converge, then their sum $\sum (a_n + b_n)$ converges as well, and
$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n.$$
2. If $\sum a_n$ and $\sum c_n$ both converge, and
$$\sum_{n=1}^{k} a_n \leq \sum_{n=1}^{k} b_n \leq \sum_{n=1}^{k} c_n \text{ for all k,}$$
then $\sum b_n$ converges as well.

**Exercise 2.1**
If $a_n \geq 0$ for all $n \in \mathbb{N}$, then $\sum a_n$ converges if and only if the corresponding sequence of partial sums $(s_k)$ is bounded.

**Task 2.2**
Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Hint: Show that the partial sums satisfy $s_k \leq 2 - \dfrac{1}{k}$.

This implies that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2$. Euler showed that the limit is actually equal to $\dfrac{\pi^2}{6} \approx 1.64493$.

**Task 2.3**
Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (= does not converge).

Hint: Show that the partial sums satisfy $s_{2^k} \geq 1 + \dfrac{k}{2}$.

**Exercise 2.4**
The series $\sum a_n$ converges if and only if for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that whenever $m > n \geq N$ it follows that

$$|a_{n+1} + a_{n+2} + \cdots a_m| < \varepsilon.$$

**Exercise 2.5**
If $\sum a_n$ converges, then $(a_n)$ converges to 0.

Note that by the example in Task 2.3 the converse does not hold.

**Exercise 2.6**
Show: If the series $\sum_{n=1}^{\infty} |a_n|$ converges, so does $\sum_{n=1}^{\infty} a_n$.

If $\sum_{n=1}^{\infty} |a_n|$ converges, we say that $\sum_{n=1}^{\infty} a_n$ *converges absolutely*. If on the other hand, $\sum_{n=1}^{\infty} a_n$ converges while $\sum_{n=1}^{\infty} |a_n|$ diverges, we say that $\sum_{n=1}^{\infty} a_n$ *converges conditionally*.

**Task 2.7**
Show that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

converges conditionally.

The example above is a special case of the next task:

**Task 2.8**
Suppose the sequence $(a_n)$ satisfies

1. $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$, and
2. the sequence $(a_n)$ converges to 0,

then $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

Given a series $\sum_{n=1}^{\infty} a_n$, we say the series $\sum_{n=1}^{\infty} b_n$ is a *rearrangement* of $\sum_{n=1}^{\infty} a_n$, if there is a bijection $\varphi : \mathbb{N} \to \mathbb{N}$ such that $b_{\varphi(n)} = a_n$ for all $n \in \mathbb{N}$.

**Task 2.9**
If the series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then any rearrangement of $\sum_{n=1}^{\infty} a_n$ converges to the same limit.

In other words: If the series is absolutely convergent, then it is "infinitely commutative." If on the other hand the series converges only conditionally, then commutativity fails in a spectacular way:

**Task 2.10**
Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges conditionally. Then for every $s \in \mathbb{R}$, there is a rearrangement $\sum_{n=1}^{\infty} b_n$ of $\sum_{n=1}^{\infty} a_n$ such that $\sum_{n=1}^{\infty} b_n$ converges to $s$.

Here are two hints to get you started on this problem:

1. Let $a_n^+ = \max\{a_n, 0\}$ and $a_n^- = \max\{-a_n, 0\}$. Thus $a_n = a_n^+ - a_n^-$ and $|a_n| = a_n^+ + a_n^-$. Observe that both series $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ do not converge.

2. The series in Task 2.7 actually converges to $\ln 2 \approx 0.693147$. Can you find a recipe how to rearrange the series so that the rearrangement converges to 1 instead?