2 Numerical Series

From this section onward the standard results from a first Analysis course are a prerequisite. For this section in particular you can (and will need to) use results about sequences.

Given a sequence $(a_n)_{n \in \mathbb{N}}$, an *infinite series* is a formal expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

The corresponding sequence of partial sums $(s_k)_{k\in\mathbb{N}}$ is defined by $s_k = a_1 + a_2 + a_3 + \cdots + a_k$.

If the sequence of partial sums converges, with limit s, we say that the series $\sum_{n=1}^{\infty} a_n$ converges and we write

$$\sum_{n=1}^{\infty} a_n = s$$

We will often write $\sum a_n$ instead of $\sum_{n=1}^{\infty} a_n$.

The following are direct consequences of the corresponding fact for sequences:

1. If the series $\sum a_n$ and $\sum b_n$ both converge, then their sum $\sum (a_n+b_n)$ converges as well, and

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n.$$

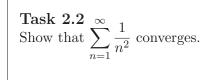
2. If $\sum a_n$ and $\sum c_n$ both converge, and

$$\sum_{n=1}^{k} a_n \le \sum_{n=1}^{k} b_n \le \sum_{n=1}^{k} c_n \text{ for all } \mathbf{k},$$

then $\sum b_n$ converges as well.

Exercise 2.1

If $a_n \ge 0$ for all $n \in \mathbb{N}$, then $\sum a_n$ converges if and only if the corresponding sequence of partial sums (s_k) is bounded.



Hint: Show that the partial sums satisfy $s_k \leq 2 - \frac{1}{k}$.

This implies that $\sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2$. Euler showed that the limit is actually equal to $\frac{\pi^2}{6} \approx 1.64493$.

Task 2.3 $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (= does not converge).

Hint: Show that the partial sums satisfy $s_{2^k} \ge 1 + \frac{k}{2}$.

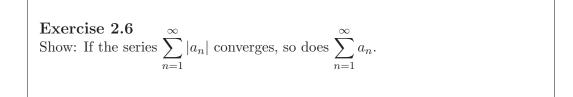
Exercise 2.4

The series $\sum a_n$ converges if and only if for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that whenever $m > n \ge N$ it follows that

$$|a_{n+1} + a_{n+2} + \cdots + a_m| < \varepsilon_1$$

Exercise 2.5 If $\sum a_n$ converges, then (a_n) converges to 0.

Note that by the example in Task 2.3 the converse does not hold.



If
$$\sum_{n=1}^{\infty} |a_n|$$
 converges, we say that $\sum_{n=1}^{\infty} a_n$ converges absolutely. If on the other hand, $\sum_{n=1}^{\infty} a_n$ converges while $\sum_{n=1}^{\infty} |a_n|$ diverges, we say that $\sum_{n=1}^{\infty} a_n$ converges conditionally.

Task 2.7 Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ converges conditionally.

The example above is a special case of the next task:

Task 2.8 Suppose the sequence (a_n) satisfies

- 1. $a_1 \ge a_2 \ge a_3 \ge \cdots \ge 0$, and
- 2. the sequence (a_n) converges to 0,

then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Given a series $\sum_{n=1}^{\infty} a_n$, we say the series $\sum_{n=1}^{\infty} b_n$ is a *rearrangement* of $\sum_{n=1}^{\infty} a_n$, if there is a bijection $\varphi : \mathbb{N} \to \mathbb{N}$ such that $b_{\varphi(n)} = a_n$ for all $n \in \mathbb{N}$.

Task 2.9 $\sum_{n=1}^{\infty} a_n$ converges absolutely, then any rearrangement of $\sum_{n=1}^{\infty} a_n$ converges to the same limit.

In other words: If the series is absolutely convergent, then it is "infinitely commutative." If on the other hand the series converges only conditionally, then commutativity fails in a spectacular way:

Task 2.10 Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges conditionally. Then for every $s \in \mathbb{R}$, there is a rearrangement $\sum_{n=1}^{\infty} b_n$ of $\sum_{n=1}^{\infty} a_n$ such that $\sum_{n=1}^{\infty} b_n$ converges to s. Here are two hints to get you started on this problem:

- 1. Let $a_n^+ = \max\{a_n, 0\}$ and $a_n^- = \max\{-a_n, 0\}$. Thus $a_n = a_n^+ a_n^-$ and $|a_n| = a_n^+ + a_n^-$. Observe that both series $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ do not converge.
- 2. The series in Task 2.7 actually converges to $\ln 2 \approx 0.693147$. Can you find a recipe how to rearrange the series so that the rearrangement converges to 1 instead?