An Introduction to Analysis on the Real Line for Classes Using Inquiry Based Learning

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Preface

Inquiry Based Learning.   These notes are designed for classes using Inquiry Based Learning, pioneered—among others—by the eminent mathematician Robert L. Moore, who taught at the University of Texas at Austin from 1920–69. The basic idea behind this method is that you can only learn how to do mathematics by doing mathematics. Here are two famous quotes attributed to R.L. Moore:

> *"There is only one math book, and this book has only one page with a single sentence: Do what you can!'"*

> *"That Student is Taught the Best Who is Told the Least."*

Prerequisites.   The only prerequisites for this course consist of knowledge of Calculus and familiarity with set notation and the Method of Proof. These prerequisites can be found in any Calculus book, and, for example, in inquiry based learning textbooks by Carol Schumacher [20] or Margie Hale [7].

Ground Rules.   Expect this course to be quite different from other mathematics courses you have taken. Here are the ground rules we will be operating under:

- These notes contain "exercises" and "tasks". You will solve these problems at home and then present the solutions in class. I will call on students at random to present "exercises"; I will call on volunteers to present solutions to the "tasks".

- When you are in the audience, you are expected to be actively engaged in the presentation. This means checking to see if every step of the presentation is clear and convincing to you, and speaking up when it is not. When there are gaps in the reasoning, the students in class will work together to fill the gaps.

- I will only serve as a moderator. My major contribution in class will consist of asking guiding and probing questions. I will also occasionally give short presentations to put topics into a wider context, or to briefly talk about additional concepts not dealt with in the notes.

- You may use only these notes and your own class notes; you are not allowed to consult other books or materials. You must not talk to anyone outside of class about the assignments. You are encouraged to collaborate with other class participants; if you do, you must acknowledge their contribution during your presentation. Exemptions from these restrictions require prior approval by the instructor.

- Your instructor is an important resource for you. I expect frequent visits from all of you during my office hours—many more visits than in a "normal" class. Among other things, you probably will want to come to my office to ask questions about concepts and assigned problems, you will probably occasionally want to show me your work before presenting it in class, and you probably will have times when you just want to talk about the frustrations you may experience.

- It is of paramount importance that we all agree to create a class atmosphere that is supportive and non-threatening to all participants. Disparaging remarks will be tolerated neither from students nor from the instructor.

Historical Perspective. This course gives an "Introduction to Analysis". After its discovery, Calculus turned out to be extremely useful in solving problems in Physics. Ad hoc justifications were used by the generation of mathematicians following Isaac Newton (1643–1727) and Gottfried Wilhelm von Leibniz (1646–1716), and even later by mathematicians such as Leonhard Euler (1707–1783), Joseph-Louis Lagrange (1736–1813) and Pierre-Simon Laplace (1749–1827). A mathematical argument given by Euler, for example, did not differ much from the kind of "explanations" you have seen in your Calculus classes.

In the first third of the nineteenth century, in particular with the publication of the essay *Théorie analytique de la chaleur* by Jean Baptiste Joseph Fourier (1768–1830) in 1822, fundamental problems with this approach of doing mathematics arose: The leading mathematicians in Europe just did not know when Fourier's ingenious method of approximating functions by trigonometric series worked, and when it failed! This led to the quest for putting the concepts of Calculus on a sound foundational basis: *What exactly does it mean for a sequence to converge? What exactly is a function? What does it mean for a function to be differentiable? When is the integral of an infinite sum of functions equal to the infinite sum of the integrals of the functions?* Etc, etc.[1]

As these fundamental questions were investigated and consequently answered by mathematicians such as Augustin Louis Cauchy (1789–1857), Bernhard Bolzano (1781–1848), G. F. Bernhard Riemann (1826–1866), Karl Weierstraß (1815–1897), and many others, the word "Analysis" became the customary term to describe this kind of "Rigorous Calculus". The progress in Analysis during the latter part of the nineteenth century and the rapid progress in the twentieth century would not have been possible without this revitalization of Calculus.

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[1]Eventually the "crisis" in Analysis also led to renewed interest into general questions concerning the nature of Mathematics. The resulting work of mathematicians and logicians such as Gottlob Frege (1848–1925), Richard Dedekind (1831–1916), Georg Cantor (1845–1918), and Bertrand Russell (1872–1970), David Hilbert (1862–1943), Kurt Gödel (1906–1978) and Paul Cohen (1934–), Luitzen Egbertus Jan Brouwer (1881–1966) and Arend Heyting (1898–1980) has fundamentally impacted all branches of mathematics and its practitioners. For a fascinating description and a very readable account of these developments and how they led to the theory of computing, see [4].

Consequently, in this course we will investigate (or in many cases revisit) the fundamental concepts in single-variable Calculus: Sequences and their convergence behavior, local and global consequences of continuity, properties of differentiable functions, integrability, and the relation between differentiability and integrability.

1 Introduction

When we want to study a subject in Mathematics, we first have to agree upon what we assume we all already understand.

In this course we will assume that we are familiar with the Real Numbers, in the sequel denoted by $\mathbb{R}$. Before we list the basic axioms the Real Numbers satisfy, we will briefly review more elementary concepts of numbers.

1.1 The Set of Natural Numbers

When we start learning Mathematics in elementary school, we live in the world of NATURAL NUMBERS, which we will denote by $\mathbb{N}$:

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$

Natural numbers are the "natural" objects to count things around us with. The first thing we learn is to add natural numbers, then later on we start to multiply.

Besides their existence, we will take the following characterization of the Natural Numbers $\mathbb{N}$ for granted throughout the course:

Axiom N1 $1 \in \mathbb{N}$.

Axiom N2 If $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$.

Axiom N3 If $n \neq m$, then $n + 1 \neq m + 1$.

Axiom N4 There is no natural number $n \in \mathbb{N}$, such that $n + 1 = 1$.

Axiom N5 If a subset $M \subseteq \mathbb{N}$ satisfies (1) $1 \in M$, and (2) $m \in M \Rightarrow m + 1 \in M$, then $M = \mathbb{N}$.

The first four axioms describe the features of the counting process: We start counting at 1, every counting number has a "successor", and counting is not "cyclic". The last axiom guarantees the **Principle of Induction**:

Task 1.1

Let $P(n)$ be a predicate with domain $\mathbb{N}$. If

1. $P(1)$ is true, and

2. Whenever $P(n)$ is true, then $P(n + 1)$ is true,

then $P(n)$ is true for all $n \in \mathbb{N}$.

1.2 Integers, Rational and Irrational Numbers

Deficiencies of the system of natural numbers start to appear when we want to divide—the quotient of two natural numbers is not necessarily a natural number, or when we want to subtract—the difference of two natural numbers is not necessarily a natural number. This leads quite naturally to two extensions of the concept of number.

The set of INTEGERS, denoted by $\mathbb{Z}$, is the set

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}.$$

The set of RATIONAL NUMBERS $\mathbb{Q}$ is defined as

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}.$$

Real numbers that are not rational are called IRRATIONAL NUMBERS. The existence of irrational numbers, first discovered by the Pythagoreans in about 520 B.C., must have come as a major surprise to Greek Mathematicians:

Task 1.2

Show that the square root of 2 is irrational. ($\sqrt{2}$ is the positive real number whose square is 2.)

1.3 Groups

Next we will put the properties of numbers and their behavior with respect to the standard arithmetic operations into a wider context by introducing the concept of an "abelian group" and, in the next section, the concept of a "field".

A set G with a binary operation $*$ is called an ABELIAN GROUP[2], if $(G, *)$ satisfies the following axioms:

G1 $*$ is a map from $G \times G$ to G.

G2 (Associativity) For all $a, b, c \in G$

$$(a * b) * c = a * (b * c)$$

G3 (Commutativity) For all $a, b \in G$

$$a * b = b * a$$

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[2]Named in honor of Niels Henrik Abel (1802–1829).

G4 (Existence of a neutral element) There is an element $n \in G$, called the neutral element of G, such that for all $a \in G$

$$a * n = a$$

G5 (Existence of inverse elements) For every $a \in G$ there exists $b \in G$, called the inverse of a, such that

$$a * b = n$$

The sets $\mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ are examples of abelian groups when endowed with the usual addition $+$. The neutral element in these cases is 0; it is customary to denote the inverse element of a by $-a$.

The sets $\mathbb{Q} \setminus \{0\} = \{r \in \mathbb{Q} \mid r \neq 0\}$ and $\mathbb{R} \setminus \{0\}$ also form abelian groups under the usual multiplication $\cdot$. In these cases we denote the neutral element by 1; the inverse element of a is customarily denoted by $1/a$ or by a^{-1}.

Exercise 1.3
Write down the axioms **G1–G5** explicitly for the set $\mathbb{Q} \setminus \{0\}$ with the binary operation $\cdot$ (i.e., multiplication).

Addition and multiplication of rational and real numbers interact in a reasonable manner—the following DISTRIBUTIVE LAW holds:

DL For all $a, b, c \in \mathbb{R}$
$$(a + b) \cdot c = (a \cdot c) + (b \cdot c)$$

1.4   Fields

In short, a set F together with an addition $+$ and a multiplication $\cdot$ is called a FIELD, if

F1 $(F, +)$ is an abelian group (with neutral element 0).
F2 $(F \setminus \{0\}, \cdot)$ is an abelian group (with neutral element 1).
F3 For all $a, b, c \in F$: $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$.

The set of rational numbers and the set of real numbers are examples of fields.

Another example of a field is the set of complex numbers $\mathbb{C}$:

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

Addition and multiplication of complex numbers are defined as follows:

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

and

$$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i,$$

respectively.

A field F endowed with a relation $\leq$ is called an ORDERED FIELD if

O1 (Antisymmetry) For all $x, y \in F$

$$x \leq y \text{ and } y \leq x \text{ implies } x = y$$

O2 (Transitivity) For all $x, y, z \in F$

$$x \leq y \text{ and } y \leq z \text{ implies } x \leq z$$

O3 For all $x, y \in F$

$$x \leq y \text{ or } y \leq x$$

O4 For all $x, y, z \in F$

$$x \leq y \text{ implies } x + z \leq y + z$$

O5 For all $x, y \in F$ and all $0 \leq z$

$$x \leq y \text{ implies } x \cdot z \leq y \cdot z$$

If $x \leq y$ and $x \neq y$, we write $x < y$. Instead of $x \leq y$, we also write $y \geq x$.

Both the rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$ form ordered fields. The complex numbers $\mathbb{C}$ cannot be ordered in such a way.

1.5   The Completeness Axiom

You probably have seen books entitled "Real Analysis" and "Complex Analysis" in the library. There are no books on "Rational Analysis".

Why? What is the main difference between the two ordered fields of $\mathbb{Q}$ and $\mathbb{R}$?—The ordered field $\mathbb{R}$ of real numbers is COMPLETE: sequences of real numbers have the following property.

C Let (a_n) be an increasing sequence of real numbers. If (a_n) is bounded from above, then (a_n) converges.

The ordered field $\mathbb{Q}$ of rational numbers, on the other hand, is **not** complete. It should therefore not surprise you that the Completeness Axiom will play a central part throughout the course! We will discuss this axiom in great detail in Section 2.3.

The complex numbers $\mathbb{C}$ also form a complete field. Section 2.6 will give an idea how to write down an appropriate completeness axiom for the field $\mathbb{C}$.

1.6   Summary: An Axiomatic System for the Set of Real Numbers

Below is a summary of the properties of the real numbers $\mathbb{R}$ we will take for granted throughout the course:

The set of real numbers $\mathbb{R}$ with its natural operations of $+, \cdot,$ and $\leq$ forms a complete ordered field. This means that the real numbers satisfy the following axioms:

Axiom 1 $+$ is a map from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$.

Axiom 2 For all $a, b, c \in \mathbb{R}$

$$(a + b) + c = a + (b + c)$$

Axiom 3 For all $a, b \in \mathbb{R}$
$$a + b = b + a$$

Axiom 4 There is an element $0 \in \mathbb{R}$ such that for all $a \in \mathbb{R}$

$$a + 0 = a$$

Axiom 5 For every $a \in \mathbb{R}$ there exists $b \in \mathbb{R}$ such that

$$a + b = 0$$

Axiom 6 $\cdot$ is a map from $\mathbb{R} \setminus \{0\} \times \mathbb{R} \setminus \{0\}$ to $\mathbb{R} \setminus \{0\}$.

Axiom 7 For all $a, b, c \in \mathbb{R} \setminus \{0\}$

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

Axiom 8 For all $a, b \in \mathbb{R} \setminus \{0\}$

$$a \cdot b = b \cdot a$$

Axiom 9 There is an element $1 \in \mathbb{R} \setminus \{0\}$ such that for all $a \in \mathbb{R} \setminus \{0\}$

$$a \cdot 1 = a$$

Axiom 10 For every $a \in \mathbb{R} \setminus \{0\}$ there exists $b \in \mathbb{R} \setminus \{0\}$ such that

$$a \cdot b = 1$$

Axiom 11 For all $a, b, c \in \mathbb{R}$

$$(a + b) \cdot c = (a \cdot c) + (b \cdot c)$$

Axiom 12 For all $a, b \in \mathbb{R}$

$$a \leq b \text{ and } b \leq a \text{ implies } a = b$$

Axiom 13 For all $a, b, c \in \mathbb{R}$

$$a \leq b \text{ and } b \leq c \text{ implies } a \leq c$$

Axiom 14 For all $a, b \in \mathbb{R}$
$$a \leq b \text{ or } a \geq b$$

Axiom 15 For all $a, b, c \in \mathbb{R}$

$$a \leq b \text{ implies } a + c \leq b + c$$

Axiom 16 For all $a, b \in \mathbb{R}$ and all $c \geq 0$

$$a \leq b \text{ implies } a \cdot c \leq b \cdot c$$

Axiom 17 Let (a_n) be an increasing sequence of real numbers. If (a_n) is bounded from above, then (a_n) converges.

1.7   Maximum and Minimum

Given a non-empty set A of real numbers, a real number b is called MAXIMUM OF THE SET A, if $b \in A$ and $b \geq a$ for all $a \in A$. Similarly, a real number s is called MINIMUM OF THE SET A, if $s \in A$ and $s \leq a$ for all $a \in A$. We write $b = \max A$, and $s = \min A$.

For example, the set $\{1, 3, 2, 0, -7, \pi\}$ has minimum -7 and maximum π, the set of natural numbers $\mathbb{N}$ has 1 as its minimum, but fails to have a maximum.

Exercise 1.4
Show that a set can have at most one maximum.

Exercise 1.5
Characterize all subsets A of the set of real numbers with the property that $\min A = \max A$.

Task 1.6
Show that finite non-empty sets of real numbers always have a minimum.

1.8 The Absolute Value

The ABSOLUTE VALUE of a real number a is defined as

$$|a| = \max\{a, -a\}.$$

For instance, $|4| = 4$, $|-\pi| = \pi$. Note that the inequalities $a \le |a|$ and $-a \le |a|$ hold for all real numbers a.

The quantity $|a - b|$ measures the distance between the two real numbers a and b on the real number line; in particular $|a|$ measures the distance of a from 0.

The following result is known as the **triangle inequality**:

Exercise 1.7
For all $a, b \in \mathbb{R}$:
$$|a + b| \le |a| + |b|$$

A related result is called the **reverse triangle inequality**:

Exercise 1.8
For all $a, b \in \mathbb{R}$:
$$|a - b| \ge \Big| |a| - |b| \Big|$$

You will use both of these inequalities frequently throughout the course.

1.9   Natural Numbers and Dense Sets inside the Real Numbers

In the sequel, we will also assume the following axiom for the Natural Numbers, even though it can be deduced from the Completeness Axiom of the Real Numbers (see Optional Task 2.1):

Axiom N6 For every positive real number $s \in \mathbb{R}$, $s > 0$, there is a natural number $n \in \mathbb{N}$ such that $n - 1 \leq s < n$.

Exercise 1.9
Show that for every positive real number r, there is a natural number n, such that
$$0 < \frac{1}{n} < r.$$

We say that a set A of real numbers is DENSE in $\mathbb{R}$, if for all real numbers $x < y$ there is an element $a \in A$ satisfying $x < a < y$.

Task 1.10
The set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$.

Task 1.11
The set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

2 Sequences and Accumulation Points

2.1 Convergent Sequences

Formally, a SEQUENCE OF REAL NUMBERS is a function $\varphi : \mathbb{N} \to \mathbb{R}$. For instance the function $\varphi(n) = \dfrac{1}{n^2}$ for all $n \in \mathbb{N}$ defines a sequence[3]. It is customary, though, to write sequences by listing their terms such as

$$1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \dots$$

or by writing them in the form $(a_n)_{n \in \mathbb{N}}$; so in our particular example we could write the sequence also as $\left(\dfrac{1}{n^2} \right)_{n \in \mathbb{N}}$. Note that the elements of a sequence come in a natural order. For instance $\dfrac{1}{49}$ is the 7th element of the sequence $\left(\dfrac{1}{n^2} \right)_{n \in \mathbb{N}}$.

Exercise 2.1

Let $(a_n)_{n \in \mathbb{N}}$ denote the sequence of prime numbers in their natural order. What is a_5?[4]

Exercise 2.2

Write the sequence $0, 1, 0, 2, 0, 3, 0, 4, \dots$ as a function $\varphi : \mathbb{N} \to \mathbb{R}$.

We say that a sequence (a_n) IS CONVERGENT, if there is a real number a, such that for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$,

$$|a_n - a| < \varepsilon.$$

The number a is called LIMIT of the sequence (a_n). We also say in this case that the sequence $(a_n)_{n \in \mathbb{N}}$ CONVERGES TO a.

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[3]See Figure 19 on page 74 for a pronunciation guide of Greek letters.
[4]"a_5" is pronounced "a sub 5".

Exercise 2.3

Spend some quality time studying Figure 1 on the next page. Explain how the pictures and the parts in the definition correspond to each other. Also reflect on how the "rigorous" definition above relates to your prior understanding of what it means for a sequence to converge.

A sequence, which fails to converge, is called DIVERGENT. Figure 2 on page 12 gives an example.

Exercise 2.4

1. Write down formally (using ε-N language) what it means that a given sequence $(a_n)_{n \in \mathbb{N}}$ does not converge to the real number a.

2. Similarly, write down what it means for a sequence to diverge.

Exercise 2.5

Show that the sequence $a_n = \dfrac{(-1)^n}{\sqrt{n}}$ converges to 0.

Exercise 2.6

Show that the sequence $a_n = 1 - \dfrac{1}{n^2 + 1}$ converges to 1.

The first general result below establishes that limits are unique.

Task 2.7

Show: If a sequence converges to two real numbers a and b, then $a = b$.

One way to proceed is to assume that the sequence converges to two numbers a and b with $a \neq b$. Then one tries to derive a contradiction, since far out sequence terms must be simultaneously close to both a and b.
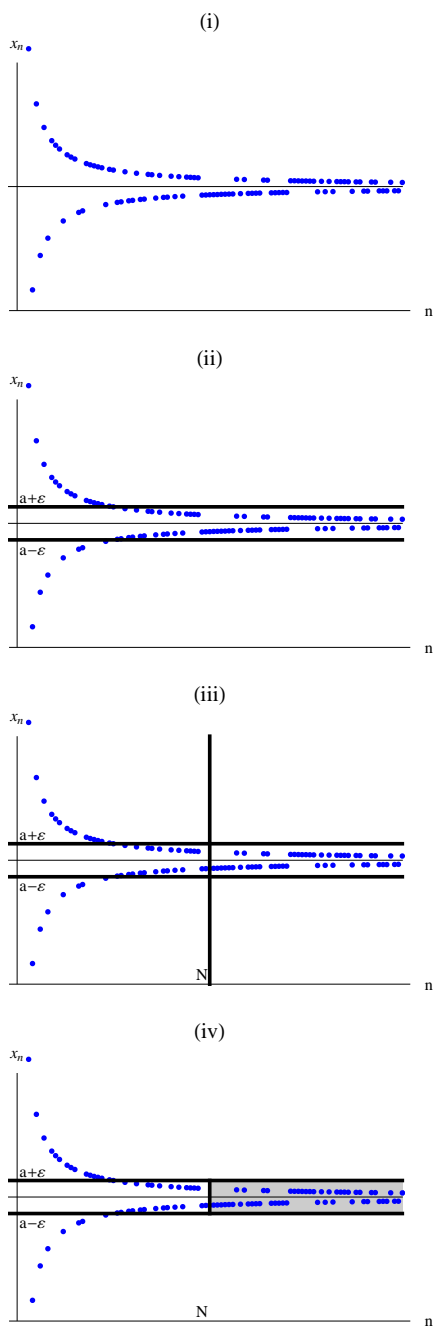
Figure 1: (i) A sequence (x_n) converges to the limit a if ... (ii) ... for all $\varepsilon > 0$ ... (iii) ... there is an $N \in \mathbb{N}$, such that ... (iv) ... $|x_n - a| < \varepsilon$ for all $n \geq N$
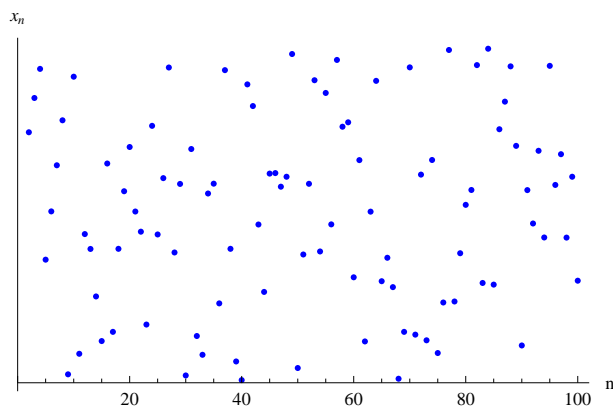
Figure 2: A divergent sequence

We say that a set S of real numbers is BOUNDED if there are real numbers m and M such that

$$m \leq s \leq M$$

holds for all $s \in S$.

A sequence (a_n) is called BOUNDED if its range

$$\{a_n \mid n \in \mathbb{N}\}$$

is a bounded set.

Exercise 2.8

Give an example of a bounded sequence which does not converge.

Task 2.9

Every convergent sequence is bounded.

Consequently, boundedness is necessary for convergence of a sequence, but is not sufficient to ensure that a sequence is convergent.

2.2   Arithmetic of Converging Sequences

The following results deal with the "arithmetic" of convergent sequences.

Task 2.10

If the sequence (a_n) converges to a, and the sequence (b_n) converges to b, then the sequence $(a_n + b_n)$ is also convergent and its limit is $a + b$.

Task 2.11

If the sequence (a_n) converges to a, and the sequence (b_n) converges to b, then the sequence $(a_n \cdot b_n)$ is also convergent and its limit is $a \cdot b$.

Task 2.12

Let (a_n) be a sequence converging to $a \neq 0$. Then there are a $\delta > 0$ and an $M \in \mathbb{N}$ such that $|a_m| > \delta$ for all $m \geq M$.

Task 2.12 is useful to prove:

Task 2.13

Let the sequence (b_n) with $b_n \neq 0$ for all $n \in \mathbb{N}$ converge to $b \neq 0$. Then the sequence $\left(\dfrac{1}{b_n}\right)$ is also convergent and its limit is $\dfrac{1}{b}$.

Task 2.14

Let (a_n) be a sequence converging to a. If $a_n \geq 0$ for all $n \in \mathbb{N}$, then $a \geq 0$.

2.3   Monotone Sequences

Let A be a non-empty set of real numbers. We say that A is BOUNDED FROM ABOVE if there is an $M \in \mathbb{R}$ such that $a \leq M$ for all $a \in A$. The number M is then called an UPPER BOUND for A. Similarly, we say that A is BOUNDED FROM BELOW if there is an $m \in \mathbb{R}$ such that $a \geq m$ for all $a \in A$. The number m is then called a LOWER BOUND for A.

A sequence is BOUNDED FROM ABOVE (BOUNDED FROM BELOW), if its range

$$\{a_n \mid n \in \mathbb{N}\}$$

is bounded from above (bounded from below).

A sequence (a_n) is called INCREASING if $a_m \leq a_n$ for all $m < n \in \mathbb{N}$. It is called STRICTLY INCREASING if $a_m < a_n$ for all $m < n \in \mathbb{N}$.

Analogously, a sequence (a_n) is called DECREASING if $a_m \geq a_n$ for all $m < n \in \mathbb{N}$. It is called STRICTLY DECREASING if $a_m > a_n$ for all $m < n \in \mathbb{N}$.

A sequence which is increasing or decreasing is called MONOTONE.

The following axiom is a fundamental property of the real numbers. It establishes that bounded monotone sequences are convergent. Most results in Analysis depend on this fundamental axiom.

> **Completeness Axiom of the Real Numbers.** Let (a_n) be an increasing bounded sequence. Then (a_n) converges.

The same result holds of course if one replaces "increasing" by "decreasing". (Can you prove this?)

Note that an increasing sequence is always bounded from below, while a decreasing sequence is always bounded from above.

Task 2.15

Let $a_1 = 1$ and $a_{n+1} = \sqrt{2a_n + 1}$ for all $n \in \mathbb{N}$. Show that the sequence (a_n) converges.

Once we know that the sequence converges, we can find its limit as follows: Let

$$L = \lim_{n \to \infty} a_n.$$

Then $\lim_{n\to\infty} a_{n+1} = L$ as well, and therefore $L = \lim_{n\to\infty} \sqrt{2a_n + 1} = \sqrt{2L + 1}$. Since L is positive and L satisfies the equation

$$L = \sqrt{2L + 1},$$

we conclude that the limit of the sequence under consideration is equal to $1 + \sqrt{2}$.

Let A be a non-empty set of real numbers. We say that a real number s is the LEAST UPPER BOUND of A (or that s is the SUPREMUM of A), if

1. s is an upper bound of A, and

2. no number smaller than s is an upper bound for A.

We write $s = \sup A$.

Similarly, we say that a real number i is the GREATEST LOWER BOUND of A (or that i is the INFIMUM of A), if

1. i is a lower bound of A, and

2. no number greater than i is a lower bound for A.

We write $i = \inf A$.

Exercise 2.16
Show the following: If a non-empty set A of real numbers has a maximum, then the maximum of A is also the supremum of A.

An INTERVAL I is a set of real numbers with the following property:

If $x \leq y$ and $x, y \in I$, then $z \in I$ for all $x \leq z \leq y$.

In particular, the set
$$[a, b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$$
is called a CLOSED INTERVAL. Similarly, the set

$$(a, b) := \{x \in \mathbb{R} \mid a < x < b\}$$

is called an OPEN INTERVAL.

Exercise 2.17

Find the supremum of each of the following sets:

1. The closed interval $[-2, 3]$

2. The open interval $(0,2)$

3. The set $\{x \in \mathbb{Z} \mid x^2 < 5\}$

4. The set $\{x \in \mathbb{Q} \mid x^2 < 3\}$.

Exercise 2.18

Let (a_n) be an increasing bounded sequence. By the Completeness Axiom the sequence converges to some real number a. Show that its range $\{a_n \mid n \in \mathbb{N}\}$ has a supremum, and that the supremum equals a.

The previous task uses the Completeness Axiom for the Real Numbers. Note that without using the Completeness Axiom we can still obtain the following weaker result: If an increasing bounded sequence converges, then it converges to the supremum of its range.

Task 2.19

The Completeness Axiom is equivalent to the following: Every non-empty set of real numbers which is bounded from above has a supremum.

The following hints may be useful to prove the "hard" direction of this result. The three hints are independent of each other and suggest different ways in which to proceed. Assume the non-empty set S is bounded from above.

- Suppose $a \in S$ and b is an upper bound of S. Then **(a)** there is an upper bound b' of S such that $|b' - a| \leq |b - a|/2$, or **(b)** there exists $a' \in S$ such that $|b - a'| \leq |b - a|/2$.

- Show that for all $\varepsilon > 0$ there is an element $a \in S$ such that $a + \varepsilon$ is an upper bound for S.

- Show that the set of upper bounds of S is of the form $[s, \infty)$. Then show that s is the supremum of S.

Optional Task 2.1

Use the Completeness Axiom to show the following: For every positive real number $s \in \mathbb{R}$, $s > 0$, there is a natural number $n \in \mathbb{N}$ such that $n - 1 \le s < n$.

2.4 Subsequences

Recall that a sequence is a function $\varphi : \mathbb{N} \to \mathbb{R}$. Let $\psi : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function[5].

Then the sequence $\varphi \circ \psi : \mathbb{N} \to \mathbb{R}$ is called a SUBSEQUENCE of φ.

Here is an example: Suppose we are given the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \dots$$

The map $\psi(n) = 2n$ then defines the subsequence

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \dots$$

If we denote the original sequence by (a_n), and if $\psi(k) = n_k$ for all $k \in \mathbb{N}$, then we denote the subsequence by (a_{n_k}).

So, in the example above,

$$a_{n_1} = a_2 = \frac{1}{2}, a_{n_2} = a_4 = \frac{1}{4}, a_{n_3} = a_6 = \frac{1}{6}, \dots$$

Exercise 2.20

Let $(a_n)_{n \in \mathbb{N}} = \left(\frac{1}{n} \right)_{n \in \mathbb{N}}$. Which of the following sequences are subsequences of $(a_n)_{n \in \mathbb{N}}$?

1. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \dots$

---

[5]A function $\psi : \mathbb{N} \to \mathbb{N}$ is called strictly increasing if it satisfies: $\psi(n) < \psi(m)$ for all $n < m$ in $\mathbb{N}$.

2. $\dfrac{1}{2}, 1, \dfrac{1}{4}, \dfrac{1}{3}, \dfrac{1}{6}, \dfrac{1}{5} \cdots$

3. $1, \dfrac{1}{3}, \dfrac{1}{6}, \dfrac{1}{10}, \dfrac{1}{15} \cdots$

4. $1, 1, \dfrac{1}{3}, \dfrac{1}{3}, \dfrac{1}{5}, \dfrac{1}{5} \cdots$

For the subsequence examples, also find the function $\psi : \mathbb{N} \to \mathbb{N}$.

Task 2.21

If a sequence converges, then all of its subsequences converge to the same limit.

Task 2.22

Show that every sequence of real numbers has an increasing subsequence or it has a decreasing subsequence.

To prove this result, the following definition may be useful: We say that the sequence (a_n) has a PEAK at n_0 if

$$a_{n_0} \geq a_n \text{ for all } n \geq n_0.$$

This result has a beautiful generalization due to Frank P. Ramsey (1903–1930), which unfortunately requires some notation: Given an infinite subset M of $\mathbb{N}$, we denote the set of doubletons from M by

$$\mathcal{P}^{(2)}(M) := \{\{m, n\} \mid m, n \in M \text{ and } m < n\}.$$

- **Ramsey's Theorem [19].** Let $\mathcal{A}$ be an arbitrary subset of $\mathcal{P}^{(2)}(\mathbb{N})$. Then there is an infinite subset M of $\mathbb{N}$ such that either

$$\mathcal{P}^{(2)}(M) \subseteq \mathcal{A}, \quad \text{or} \quad \mathcal{P}^{(2)}(M) \cap \mathcal{A} = \emptyset.$$

You should prove Task 2.22 without using this theorem, but the result in Task 2.22 follows easily from Ramsey's Theorem: Set

$$\mathcal{A} = \{\{m, n\} \mid m < n \text{ and } a_m \leq a_n\}.$$

Does your proof of Task 2.22 actually show a slightly stronger result?

The next fundamental result is known as the **Bolzano-Weierstrass Theorem.**

Task 2.23

Every bounded sequence of real numbers has a convergent subsequence.

Exercise 2.24

Let $a < b$. Every sequence contained in the interval $[a, b]$ has a subsequence that converges to an element in $[a, b]$.

Task 2.25

Suppose the sequence (a_n) does **not** converge to the real number L. Then there is an $\varepsilon > 0$ and a subsequence (a_{n_k}) of (a_n) such that

$$|a_{n_k} - L| \geq \varepsilon \text{ for all } k \in \mathbb{N}.$$

We conclude this section with a rather strange result: it establishes convergence of a bounded sequence without ever showing any convergence at all.

Task 2.26

Let (a_n) be a **bounded** sequence. Suppose all of its **convergent** subsequences converge to the same limit a. Then (a_n) itself converges to a.

2.5   Limes Inferior and Limes Superior*

Let (a_n) be a bounded sequence of real numbers. We define the LIMES INFERIOR[6] and LIMES SUPERIOR of the sequence as

$$\liminf_{n \to \infty} a_n := \lim_{k \to \infty} \left(\inf\{a_n \mid n \geq k\} \right),$$

---

[6] "limes" means limit in Latin.

and
$$\limsup_{n\to\infty} a_n := \lim_{k\to\infty} \left(\sup\{a_n \mid n \geq k\}\right).$$

Optional Task 2.2

Explain why the numbers $\liminf\limits_{n\to\infty} a_n$ and $\limsup\limits_{n\to\infty} a_n$ are well-defined[7] for every bounded sequence (a_n).

One can define the notions of $\limsup$ and $\liminf$ without knowing what a limit is:

Optional Task 2.3

Show that the limes inferior and the limes superior can also be defined as follows:

$$\liminf_{n\to\infty} a_n := \sup\left\{\inf\{a_n \mid n \geq k\} \mid k \in \mathbb{N}\right\},$$

and

$$\limsup_{n\to\infty} a_n := \inf\left\{\sup\{a_n \mid n \geq k\} \mid k \in \mathbb{N}\right\}.$$

Optional Task 2.4

Show that a bounded sequence (a_n) converges if and only if

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$

Optional Task 2.5

Let (a_n) be a bounded sequence of real numbers. Show that (a_n) has a subsequence that converges to $\limsup\limits_{n\to\infty} a_n$.

---

[7]An object is well-defined if it exists and is uniquely determined.

Optional Task 2.6

Let (a_n) be a bounded sequence of real numbers, and let (a_{n_k}) be one of its converging subsequences. Show that

$$\liminf_{n\to\infty} a_n \leq \lim_{k\to\infty} a_{n_k} \leq \limsup_{n\to\infty} a_n.$$

2.6 Cauchy Sequences

A sequence $(a_n)_{n\in\mathbb{N}}$ is called a CAUCHY SEQUENCE[8], if for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$ with $m \geq N$ and $n \geq N$,

$$|a_m - a_n| < \varepsilon.$$

Informally speaking: a sequence is convergent, if far out all terms of the sequence are close to the limit; a sequence is a Cauchy sequence, if far out all terms of the sequence are close to each other.

We will establish in this section that a sequence converges if and only if it is a Cauchy sequence.

You may wonder why we bother to explore the concept of a Cauchy sequence when it turns out that Cauchy sequences are nothing else but convergent sequences. Answer: You can't show directly that a sequence converges without knowing its limit a priori. The concept of a Cauchy sequence on the other hand allows you to show convergence without knowing the limit of the sequence in question! This will nearly always be the situation when you study series of real numbers. The "Cauchy criterion" for series turns out to be one of most widely used tools to establish convergence of series.

Exercise 2.27

Every convergent sequence is a Cauchy sequence.

Exercise 2.28

Every Cauchy sequence is bounded.

---

[8]Named in honor of Augustin Louis Cauchy (1789–1857)

Task 2.29

If a Cauchy sequence has a converging subsequence with limit a, then the Cauchy sequence itself converges to a.

Task 2.30

Every Cauchy sequence is convergent.

Optional Task 2.7

Show that the following three versions of the Completeness Axiom are equivalent:

1. Every increasing bounded sequence of real numbers converges.

2. Every non-empty set of real numbers which is bounded from above has a supremum.

3. Every Cauchy sequence of real numbers converges.

From an abstract point of view, our course in "Analysis on the Real Line" hinges on two concepts:

- We can measure the **distance** between real numbers. More precisely we can measure "small" distances: for instance, we have constructs such as "$|a_n - a| < \varepsilon$" measuring how "close" a_n and a are.

- We can **order** real numbers. The statement "For all $\varepsilon > 0$ there is an $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < \varepsilon$", for example, relies exclusively on our ability to order real numbers.

Thus the last two versions of the Completeness Axiom point to possible generalizations of our subject matter.

The second version uses the concept of **order** (boundedness, least upper bound), but does not mention distance. Section 2.5 gives clues how to define the limit concept in such a scenario.

The last version of the Completeness Axiom, on the other hand, requires the ability to measure small **distances**, but does not rely on order. This will be useful when defining completeness for sets such as $\mathbb{C}$ and $\mathbb{R}^n$ that cannot be ordered in the same way real numbers can.

2.7 Accumulation Points

Given $x \in \mathbb{R}$ and $\varepsilon > 0$, we say that the open interval $(x - \varepsilon, x + \varepsilon)$ forms a NEIGHBOR-HOOD OF x.

We say that a property $P(n)$ holds FOR ALL BUT FINITELY MANY $n \in \mathbb{N}$ if the set $\{n \in \mathbb{N} \mid P(n) \text{ does not hold}\}$ is finite.

Task 2.31

A sequence (a_n) converges to $L \in \mathbb{R}$ if and only if every neighborhood of L contains all but a finite number of the terms of the sequence (a_n).

The real number x is called an ACCUMULATION POINT of the set S, if every neighborhood of x contains infinitely many elements of S.

Task 2.32

The real number x is an accumulation point of the set S if and only if every neighborhood of x contains an element of S different from x.

Note that finite sets do not have accumulation points.

The following exercise provides some more examples:

Exercise 2.33

Find all accumulation points of the following sets:

1. $\mathbb{Q}$

2. $\mathbb{N}$

3. $[a, b)$

---

4. $\left\{ \dfrac{1}{n} \mid n \in \mathbb{N} \right\}$.

Exercise 2.34

1. Find a set of real numbers with exactly two accumulation points.

2. Find a set of real numbers whose accumulation points form a sequence (a_n) with $a_n \neq a_m$ for all $n \neq m$.

Task 2.35

Show that x is an accumulation point of the set S if and only if there is a sequence (x_n) of elements in S with $x_n \neq x_m$ for all $n \neq m$ such that (x_n) converges to x.

Task 2.36

Every infinite bounded set of real numbers has at least one accumulation point.

Optional Task 2.8

Characterize all infinite sets that have no accumulation points.

The next tasks in this section explore the relationship between the limit of a converging sequence and accumulation points of its range.

Optional Task 2.9

1. Find a converging sequence whose range has exactly one accumulation point.

2. Find a converging sequence whose range has no accumulation points.

3. Show that the range of a converging sequence has at most one accumulation point.

Optional Task 2.10

Suppose the sequence (a_n) is bounded and satisfies the condition that $a_m \neq a_n$ for all $m \neq n \in \mathbb{N}$. If its range $\{a_n \mid n \in \mathbb{N}\}$ has exactly one accumulation point a, then (a_n) converges to a.

The remaining tasks investigate how accumulation points behave with respect to some of the usual operations of set theory.

For any set S of real numbers, we denote by $\mathbf{A}(S)$ the set of all accumulation points of S.

Optional Task 2.11

If S and T are two sets of real numbers and if $S \subseteq T$, then $\mathbf{A}(S) \subseteq \mathbf{A}(T)$.

Optional Task 2.12

If S and T are two sets of real numbers, then

$$\mathbf{A}(S \cup T) = \mathbf{A}(S) \cup \mathbf{A}(T).$$

Optional Task 2.13

1. Let $(S_n)_{n \in \mathbb{N}}$ be a collection of sets of real numbers. Show that

$$\bigcup_{n \in \mathbb{N}} \mathbf{A}(S_n) \subseteq \mathbf{A}\left(\bigcup_{n \in \mathbb{N}} S_n\right).$$

2. Find a collection $(S_n)_{n \in \mathbb{N}}$ of sets of real numbers such that

$$\bigcup_{n \in \mathbb{N}} \mathbf{A}(S_n) \quad \text{is a proper subset of} \quad \mathbf{A}\left(\bigcup_{n \in \mathbb{N}} S_n\right).$$

Optional Task 2.14
Let S be a set of real numbers. Show that $\mathbf{A}(\mathbf{A}(S)) \subseteq \mathbf{A}(S)$.

3 Limits

3.1 Definition and Examples

Let $D \subseteq \mathbb{R}$, let $f : D \to \mathbb{R}$ be a function and let x_0 be an accumulation point of D.

We say that the LIMIT of $f(x)$ at x_0 is equal to $L \in \mathbb{R}$, if for all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon$$

whenever $x \in D$ and $0 < |x - x_0| < \delta$.

In this case we write $\lim\limits_{x \to x_0} f(x) = L$.

Note that—by design—the existence of the limit (and L itself) does not depend on what happens when $x = x_0$, but only on what happens "close" to x_0.
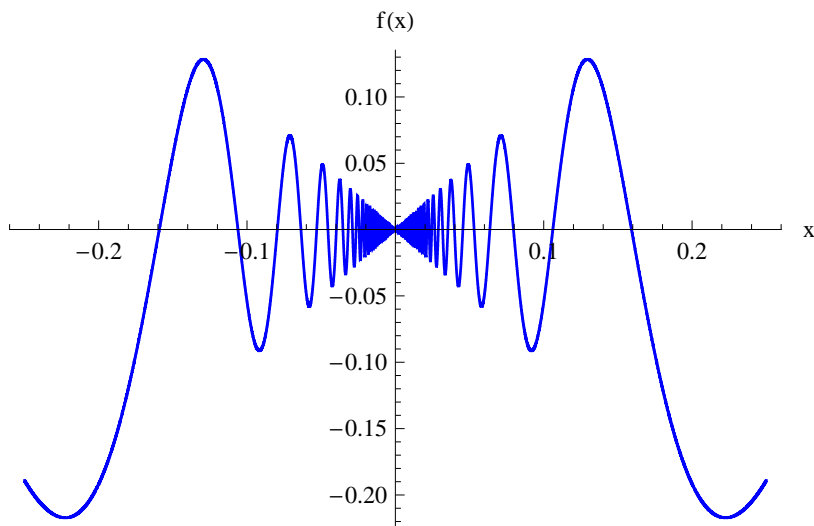
Figure 3: The graph of $x \sin(1/x)$

Exercise 3.1
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0, \ x \in \mathbb{R} \\ 0, & \text{if } x = 0 \end{cases}$$

Does $f(x)$ have a limit a $x_0 = 0$? If so, what is the limit? See Figure 3 on the page before.

The next result reduces the study of the concept of a limit of a function at a point to our earlier study of sequence convergence.

Exercise 3.2
Let $D \subseteq \mathbb{R}$, let $f : D \to \mathbb{R}$ be a function and let x_0 be an accumulation point of D. Then the following are equivalent:

1. $\lim\limits_{x \to x_0} f(x)$ exists and is equal to L.

2. Let (x_n) be any sequence of elements in D that converges to x_0, and satisfies that $x_n \neq x_0$ for all $n \in \mathbb{N}$. Then the sequence $f(x_n)$ converges to L.

Exercise 3.3
Let $D \subseteq \mathbb{R}$, let $f : D \to \mathbb{R}$ be a function and let x_0 be an accumulation point of D.

Suppose that there is an $\varepsilon > 0$ such that for all $\delta > 0$ there are $x, y \in D \setminus \{x_0\}$ satisfying $|x - y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon$. Then f does not have a limit at x_0.

Exercise 3.4
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} |x|/x, & \text{if } x \neq 0, \ x \in \mathbb{R} \\ 0, & \text{if } x = 0 \end{cases}$$

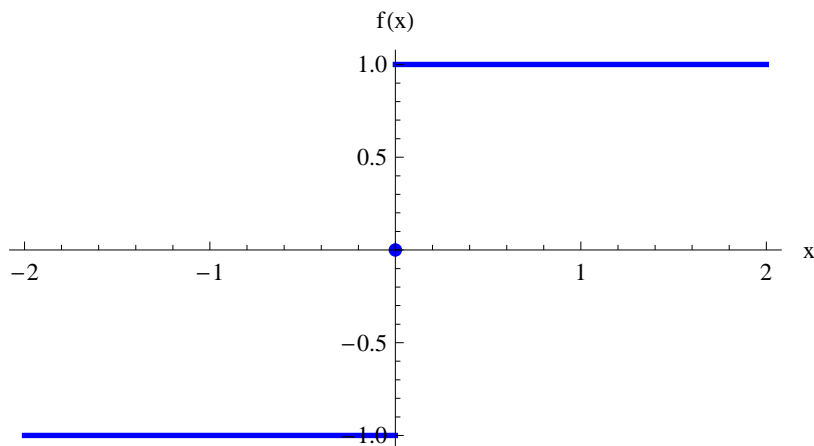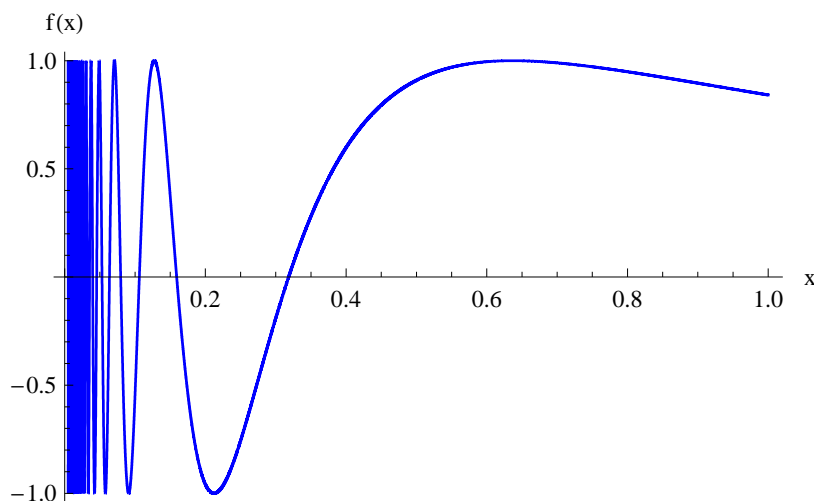Does $f(x)$ have a limit a $x_0 = 0$? If so, what is the limit? See Figure 4 on the following page.

Figure 4: The graph of the function in Exercise 3.4

Exercise 3.5

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0, \ x \in \mathbb{R} \\ 0, & \text{if } x = 0 \end{cases}$$

Does $f(x)$ have a limit a $x_0 = 0$? If so, what is the limit? See Figure 5 on the next page.

Exercise 3.6

Let $f : (0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

For which values of x_0 does $f(x)$ have a limit a x_0? What is the limit?

Figure 5: The graph of $\sin\left(1/x\right)$

Task 3.7

Let $f : (0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \dfrac{1}{q}, & \text{if } x = \dfrac{p}{q} \text{ with } p, q \text{ relatively prime positive integers} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

For which values of x_0 does $f(x)$ have a limit a x_0? What is the limit? See Figure 6 on the following page.

You may want to try the case $x_0 = 0$ first.

The result below is called the **Principle of Local Boundedness**.

Exercise 3.8

Let $D \subseteq \mathbb{R}$, let $f : D \to \mathbb{R}$ be a function and let x_0 be an accumulation point of D.

If $f(x)$ has a limit at x_0, then there is a $\delta > 0$ and an $M > 0$ such that

$$|f(x)| \le M \text{ for all } x \in (x_0 - \delta, x_0 + \delta) \cap D.$$
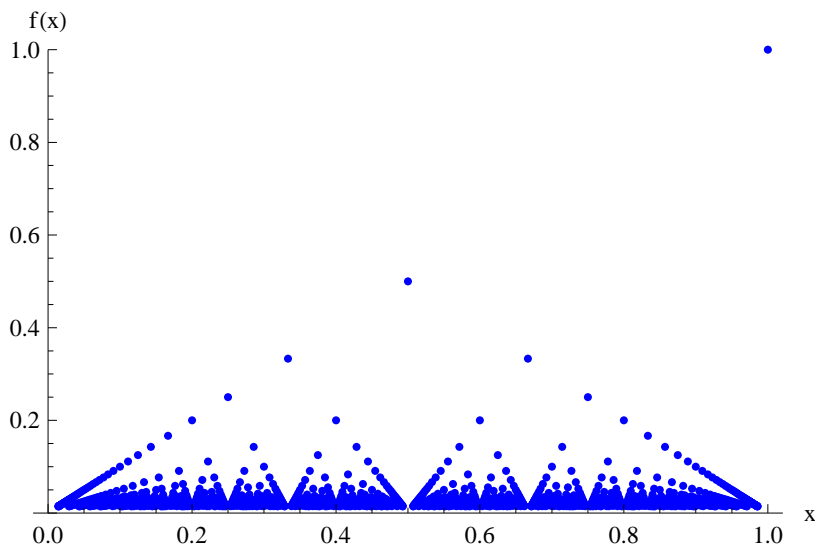
Figure 6: The graph of the function in Task 3.7

3.2   Arithmetic of Limits*

Optional Task 3.1

Let $D \subseteq \mathbb{R}$, let $f, g : D \to \mathbb{R}$ be functions and let x_0 be an accumulation point of D.

If $\lim\limits_{x \to x_0} f(x) = L$ and $\lim\limits_{x \to x_0} g(x) = M$, then the sum $f + g$ has a limit at x_0, and $\lim\limits_{x \to x_0} (f + g)(x) = L + M$.

Optional Task 3.2

Let $D \subseteq \mathbb{R}$, let $f, g : D \to \mathbb{R}$ be functions and let x_0 be an accumulation point of D.

If $\lim\limits_{x \to x_0} f(x) = L$ and $\lim\limits_{x \to x_0} g(x) = M$, then the product $f \cdot g$ has a limit at x_0, and $\lim\limits_{x \to x_0} (f \cdot g)(x) = L \cdot M$.

Optional Task 3.3

Let $D \subseteq \mathbb{R}$, let $f : D \to \mathbb{R}$ be a function and let x_0 be an accumulation point of D. Assume additionally that $f(x) \neq 0$ for all $x \in D$.

If $\lim\limits_{x \to x_0} f(x) = L$ and if $L \neq 0$, then the reciprocal function $1/f : D \to \mathbb{R}$ has a limit at x_0, and $\lim\limits_{x \to x_0} \dfrac{1}{f(x)} = \dfrac{1}{L}$.

3.3   Monotone Functions*

Let $a < b$ be real numbers. A function $f : [a, b] \to \mathbb{R}$ is called INCREASING on $[a, b]$, if $x < y$ implies $f(x) \leq f(y)$ for all $x, y \in [a, b]$. It is called STRICTLY INCREASING on $[a, b]$, if $x < y$ implies $f(x) < f(y)$ for all $x, y \in [a, b]$.

Similarly, a function $f : [a, b] \to \mathbb{R}$ is called DECREASING on $[a, b]$, if $x < y$ implies $f(x) \geq f(y)$ for all $x, y \in [a, b]$. It is called STRICTLY DECREASING on $[a, b]$, if $x < y$ implies $f(x) > f(y)$ for all $x, y \in [a, b]$.

A function $f : [a, b] \to \mathbb{R}$ is called MONOTONE on $[a, b]$ if it is increasing on $[a, b]$ or it is decreasing on $[a, b]$.

As we have seen in the last section, a function can fail to have limits for various reasons. Monotone functions, on the other hand, are easier to understand: a monotone function fails to have a limit at a point if and only if it "jumps" at that point. The next task makes this precise.

Optional Task 3.4

Let $a < b$ be real numbers and let $f : [a, b]$ be an **increasing** function. Let $x_0 \in (a, b)$. We define

$$L(x_0) = \sup\{f(y) \mid y \in [a, x_0)\}$$

and

$$U(x_0) = \inf\{f(y) \mid y \in (x_0, b]\}$$

Then $f(x)$ has a limit at x_0 if and only if $U(x_0) = L(x_0)$. In this case

$$U(x_0) = L(x_0) = f(x_0) = \lim\limits_{x \to x_0} f(x).$$

Optional Task 3.5

Under the assumptions of the previous task, state and prove a result discussing the existence of a limit at the endpoints a and b.

Optional Task 3.6

Let $a < b$ be real numbers and let $f : [a, b]$ be an increasing function. Show that the set

$$\{y \in [a, b] \mid f(x) \text{ does not have a limit at } y\}$$

is finite or countable[9].

You may want to show first that the set

$$D_n := \{y \in (a, b) \mid (U(y) - L(y)) > 1/n\}$$

is finite for all $n \in \mathbb{N}$.

Let us look at an example of an increasing function with countably many "jumps": Let $g : [0, 1] \to [0, 1]$ be defined as follows:

$$g(x) = \begin{cases} 0 & , \text{ if } x = 0 \\ \dfrac{1}{n} & , \text{ if } x \in \left(\dfrac{1}{n+1}, \dfrac{1}{n} \right] \text{ for some } n \in \mathbb{N} \end{cases}$$

Figure 7 on the next page shows the graph of $g(x)$. Note that the function is well defined, since

$$\bigcup_{n \in \mathbb{N}} \left(\frac{1}{n+1}, \frac{1}{n} \right] = (0, 1],$$

and

$$\left(\frac{1}{m+1}, \frac{1}{m} \right] \cap \left(\frac{1}{n+1}, \frac{1}{n} \right] = \emptyset$$

for all $m, n \in \mathbb{N}$ with $m \neq n$.

---

[9]A set is called COUNTABLE, if all of its elements can be arranged as a sequence $y_1, y_2, y_3, \dots$ with $y_i \neq y_j$ for all $i \neq j$.
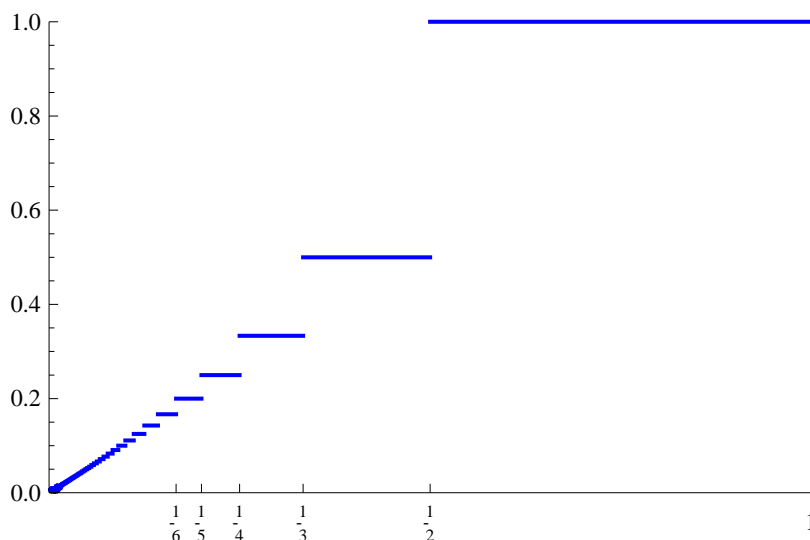
Figure 7: The graph of a function with countable many "jumps"

Optional Task 3.7
Show the following:

1. The function $g(x)$ defined above fails to have a limit at all points in the set
   $$D := \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}.$$

2. The function $g(x)$ has a limit at all points in the complement $[0, 1] \setminus D$.

4 Continuity

4.1 Definition and Examples

Let D be a set of real numbers and $x_0 \in D$. A function $f : D \to \mathbb{R}$ is said to be CONTINUOUS at x_0 if the following holds: For all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in D$ with

$$|x - x_0| < \delta,$$

we have that

$$|f(x) - f(x_0)| < \varepsilon.$$

If the function is continuous at all $x_0 \in D$, we simply say that $f : D \to \mathbb{R}$ is continuous on D.

Compare this definition of continuity to the earlier definition of having a limit. For continuity, we want to ensure that the behavior of the function close to the point x_0 nicely interacts with the behavior of the function at the point in question itself; thus we require that x_0 lies in the domain D, and that the "limit" equals $f(x_0)$. Note also that we do no longer require in the definition of continuity that x_0 is an accumulation point of D.

Exercise 4.1

Let D be a set of real numbers and $x_0 \in D$ be an accumulation point of D. Then the function $f : D \to \mathbb{R}$ is continuous at x_0 if and only if $\lim_{x \to x_0} f(x) = f(x_0)$.

Exercise 4.2

Let D be a set of real numbers and $x_0 \in D$. Assume also that x_0 is not an accumulation point of D. Then the function $f : D \to \mathbb{R}$ is continuous at x_0.

Optional Task 4.1

Let D be a set of real numbers and $x_0 \in D$. A function $f : D \to \mathbb{R}$ is continuous at x_0 if and only if for all sequences (x_n) in D converging to x_0, the sequence $(f(x_n))$ converges to $f(x_0)$.

Exercise 4.3
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \left\{ \begin{array}{ll} |x|, & \text{if } x \in \mathbb{Q} \\ x^2, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{array} \right.$$

For which values of x_0 is $f(x)$ continuous?

Exercise 4.4
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \left\{ \begin{array}{ll} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0, \ x \in \mathbb{R} \\ 0, & \text{if } x = 0 \end{array} \right.$$

Is $f(x)$ continuous at $x_0 = 0$? See Figure 3 on page 27.

Exercise 4.5
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \left\{ \begin{array}{ll} \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0, \ x \in \mathbb{R} \\ 0, & \text{if } x = 0 \end{array} \right.$$

Is $f(x)$ continuous at $x_0 = 0$?

See Figure 5 on page 30.

Exercise 4.6
Let $f : (0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \left\{ \begin{array}{ll} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{array} \right.$$

For which values of x_0 is $f(x)$ continuous?

Exercise 4.7

Let $f : (0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \dfrac{1}{q}, & \text{if } x = \dfrac{p}{q} \text{ with } p, q \text{ relatively prime positive integers} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

For which values of x_0 is $f(x)$ continuous? See Figure 6 on page 31.

It is interesting to note that in the late 1890s René-Louis Baire (1874–1932) proved a beautiful result which implies that there are no functions on the real line that are continuous at all rational numbers and discontinuous at all irrational numbers.

4.2 Combinations of Continuous Functions

Optional Task 4.2

Let $D \subseteq \mathbb{R}$, and let $f, g : D \to \mathbb{R}$ be functions continuous at $x_0 \in D$. Then $f + g : D \to \mathbb{R}$ is continuous at x_0.

Optional Task 4.3

Let $D \subseteq \mathbb{R}$, and let $f, g : D \to \mathbb{R}$ be functions continuous at $x_0 \in D$. Then $f \cdot g : D \to \mathbb{R}$ is continuous at x_0.

Optional Task 4.4

Polynomials are continuous on $\mathbb{R}$.

Optional Task 4.5

Let $D \subseteq \mathbb{R}$, and let $f : D \to \mathbb{R}$ be a function continuous at $x_0 \in D$. Assume additionally that $f(x) \neq 0$ for all $x \in D$. Then $\dfrac{1}{f} : D \to \mathbb{R}$ is continuous at x_0.

Task 4.8

Let $D, E \subseteq \mathbb{R}$, and let $f : D \to \mathbb{R}$ be a function continuous at $x_0 \in D$. Assume $f(D) \subseteq E$. Suppose $g : E \to \mathbb{R}$ is a function continuous at $f(x_0)$. Then the composition $g \circ f : D \to \mathbb{R}$ is continuous at x_0.

4.3 Uniform Continuity

We say that a function $f : D \to \mathbb{R}$ is UNIFORMLY CONTINUOUS on D if the following holds: For all $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $x, y \in D$ satisfy

$$|x - y| < \delta,$$

then

$$|f(x) - f(y)| < \varepsilon.$$

Exercise 4.9

If $f : D \to \mathbb{R}$ is uniformly continuous on D, then f is continuous on D. What is the difference between continuity and uniform continuity?

Exercise 4.10

Let $f : (0, 1) \to \mathbb{R}$ be defined by $f(x) = \dfrac{1}{x}$. Show that f is not uniformly continuous on $(0, 1)$.

Similarly one can show that the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^2$, is continuous on $\mathbb{R}$, but fails to be uniformly continuous on $\mathbb{R}$.

Task 4.11

Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$. Show that f is uniformly continuous on $[a, b]$.

Along the way, you probably want to use the Bolzano-Weierstrass Theorem (Task 2.23 on page 19) to prove this result.

In light of Exercise 4.10, the result of Task 4.11 must depend heavily on properties of the domain. It is therefore natural to ask for what domains continuity automatically implies uniform continuity. The following two tasks explore this question.

Optional Task 4.6

Let $f : \mathbb{N} \to \mathbb{R}$ be an arbitrary function. Then f is uniformly continuous on $\mathbb{N}$.

Optional Task 4.7

Let D be a set of real numbers. Give a characterization of all the domains D such that every continuous function $f : D \to \mathbb{R}$ is uniformly continuous on D. [11]

Task 4.12

Let $f : D \to \mathbb{R}$ be uniformly continuous on D. If D is a bounded subset of $\mathbb{R}$, then $f(D)$ is also bounded.

Optional Task 4.8

Let $f : D \to \mathbb{R}$ be uniformly continuous on D. If (x_n) is a Cauchy sequence in D, then $(f(x_n))$ is also a Cauchy sequence.

Note a subtle, but important difference between the conclusion of the Task above and the characterization of continuity in Exercise 4.1: Even though every Cauchy sequence of elements in D will converge to some real number, that real number will not necessarily lie in D.

Optional Task 4.9

If a function $f : (a, b) \to \mathbb{R}$ is uniformly continuous on the open interval (a, b), then it can be defined at the endpoints a and b in such a way that the extension $f : [a, b] \to \mathbb{R}$ is (uniformly) continuous on the closed interval $[a, b]$.

Thus, for instance, the function $f : (0, 1) \to \mathbb{R}$, given by $f(x) = \sin\left(\dfrac{1}{x}\right)$ is not uniformly continuous on $(0, 1)$.

It is often easier to show uniform continuity by establishing the following stronger condition:

A function $f : D \to \mathbb{R}$ is called a LIPSCHITZ FUNCTION on D if there is an $M > 0$ such that for all $x, y \in D$
$$|f(x) - f(y)| \le M|x - y|$$

Exercise 4.13
Let $f : D \to \mathbb{R}$ be a Lipschitz function on D. Then f is uniformly continuous on D.

Task 4.14
Show: The function $f(x) = \sqrt{x}$ is uniformly continuous on the interval $[0, 1]$, but it is not a Lipschitz function on the interval $[0, 1]$.

4.4   Continuous Functions on Closed Intervals

The major goal of this section is to show that the continuous image of a closed bounded interval is a closed bounded interval.

We say a function $f : D \to \mathbb{R}$ is BOUNDED, if there exists an $M \in \mathbb{R}$ such that $|f(x)| \le M$ for all $x \in D$.

Exercise 4.15
Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$. Then f is bounded on $[a, b]$.

Optional Task 4.10
Let D be a set of real numbers. Give a characterization of all the domains D such that every continuous function $f : D \to \mathbb{R}$ is automatically bounded on D.   [9]

We say that the function $f : D \to \mathbb{R}$ has an ABSOLUTE MAXIMUM if there exists an $x_0 \in D$ such that $f(x) \le f(x_0)$ for all $x \in D$. Similarly, $f : D \to \mathbb{R}$ has an ABSOLUTE MINIMUM if there exists an $x_0 \in D$ such that $f(x) \ge f(x_0)$ for all $x \in D$.

We can improve upon the result of Exercise 4.15 as follows:

Task 4.16
Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$. Then f has an absolute maximum (and an absolute minimum) on $[a, b]$.

The next result is called the **Intermediate Value Theorem**.
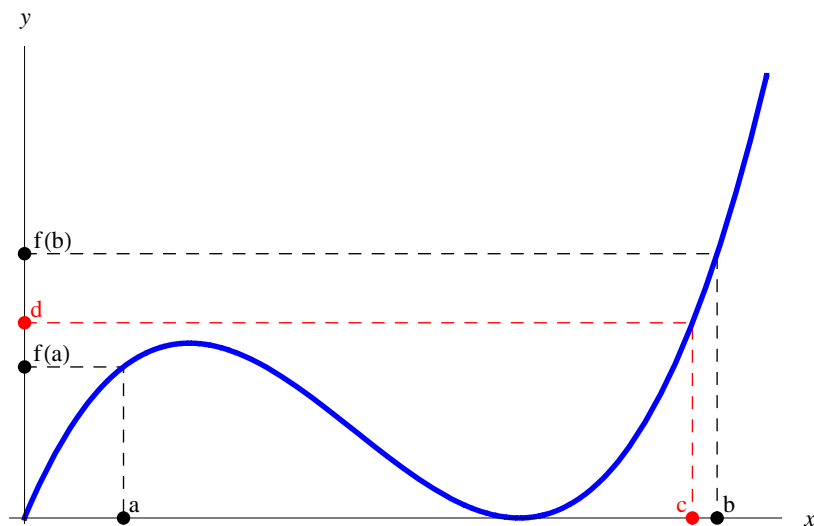
Figure 8: The Intermediate Value Theorem

Here the interval I can be any interval. Also: If $x > y$, we understand the interval (x, y) to be the interval (y, x).

Task 4.17
Let $f : I \to \mathbb{R}$ be a continuous function on the interval I. Let $a, b \in I$. If $d \in (f(a), f(b))$, then there is a real number $c \in (a, b)$ such that $f(c) = d$. See Figure 8.

A continuous function maps a closed bounded interval onto a closed bounded interval:

Task 4.18

Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$. Then $f([a, b]) := \{f(x) \mid x \in [a, b]\}$ is also a closed bounded interval.

Task 4.19

Let $f : [a, b] \to \mathbb{R}$ be strictly increasing (or decreasing, resp.) and continuous on $[a, b]$. Show that f has an inverse on $f([a, b])$, which is strictly increasing (or decreasing, resp.) and continuous.

Task 2.26 may be helpful to prove this result.

Task 4.20

Show that $\sqrt{x} : [0, \infty) \to \mathbb{R}$ is continuous on $[0, \infty)$.

---

5 The Derivative

5.1 Definition and Examples

Let D be a set of real numbers and let $x_0 \in D$ be an accumulation point of D. The function $f : D \to \mathbb{R}$ is said to be DIFFERENTIABLE at x_0, if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exists.}$$

In this case, we call the limit above the DERIVATIVE of f at x_0 and write

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Exercise 5.1

Use the definition above to show that $\sqrt[3]{x} : \mathbb{R} \to \mathbb{R}$ is differentiable at $x_0 = -27$ and that its derivative at $x_0 = -27$ equals $\frac{1}{27}$.

Exercise 5.2

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0, \ x \in \mathbb{R} \\ 0, & \text{if } x = 0 \end{cases}$$

Is $f(x)$ differentiable at $x_0 = 0$? See Figure 3 on page 27.

Exercise 5.3

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0, \ x \in \mathbb{R} \\ 0, & \text{if } x = 0 \end{cases}$$

Is $f(x)$ differentiable at $x_0 = 0$? Using your Calculus knowledge, compute the derivative at points $x_0 \neq 0$. Is the derivative continuous at $x_0 = 0$? See Figure 9 on the following page.
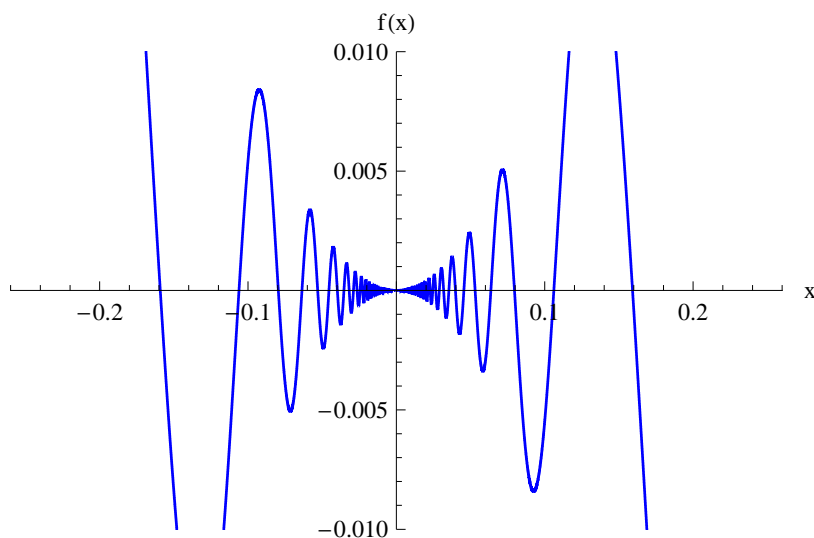
Figure 9: The graph of $x^2 \sin\left(1/x\right)$

5.2   Techniques of Differentiation

Exercise 5.4
Suppose $f : D \to \mathbb{R}$ is differentiable at $x_0 \in D$. Show that f is continuous at x_0.

Exercise 5.5
Give an example of a function with a point at which f is continuous, but not differentiable.

Exercise 5.6
Let $f, g : D \to \mathbb{R}$ be differentiable at $x_0 \in D$. Then the function $f+g$ is differentiable at x_0, with $(f + g)'(x_0) = f'(x_0) + g'(x_0)$.

Next come some of the "Calculus Classics", beginning with the "Product Rule":

Task 5.7

Let $f, g : D \to \mathbb{R}$ be differentiable at $x_0 \in D$. Then the function $f \cdot g$ is differentiable at x_0, with

$$(f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0).$$

In particular, if $c \in \mathbb{R}$, then

$$(c \cdot f)'(x_0) = c \cdot f'(x_0).$$

Exercise 5.8

Show that polynomials are differentiable everywhere.

Compute the derivative of a polynomial of the form

$$P(x) = \sum_{k=0}^{n} a_k x^k.$$

Optional Task 5.1

State and prove the "Quotient Rule".

Optional Task 5.2

State and prove the "Chain Rule".

5.3   The Mean-Value Theorem and its Applications

Let D be a subset of $\mathbb{R}$, and let $f : D \to \mathbb{R}$ be a function. We say that f has a LOCAL MAXIMUM at $x_0 \in D$, if there is a neighborhood U of x_0, such that

$$f(x) \leq f(x_0) \text{ for all } x \in U.$$

Similarly, we say that f has a LOCAL MINIMUM at $x_0 \in D$, if there is a neighborhood U of x_0, such that

$$f(x) \geq f(x_0) \text{ for all } x \in U.$$

The next result is commonly known as the **First Derivative Test**. Note that this only works for $x_0 \in (a, b)$, not if x_0 is one of the endpoints a or b.

Task 5.9

Suppose $f : [a, b] \to \mathbb{R}$ has either a local maximum or a local minimum at $x_0 \in (a, b)$. If f is differentiable at x_0, then $f'(x_0) = 0$.

Task 5.10

Suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b).

If $f(a) = f(b) = 0$, then there exists a $c \in (a, b)$ with $f'(c) = 0$.

This result is usually called **Rolle's Theorem**, named after Michel Rolle (1652–1719). A much more useful version of Task 5.10 is known as the **Mean Value Theorem**:

Task 5.11

Suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b).

Then there exists a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

See Figure 10 on the next page.

Do not confuse the Mean Value Theorem with the Intermediate Value Theorem!

Nearly all properties of differentiable functions follow from the Mean Value Theorem. The exercises and tasks below are such examples of straightforward applications of the Mean-Value Theorem.
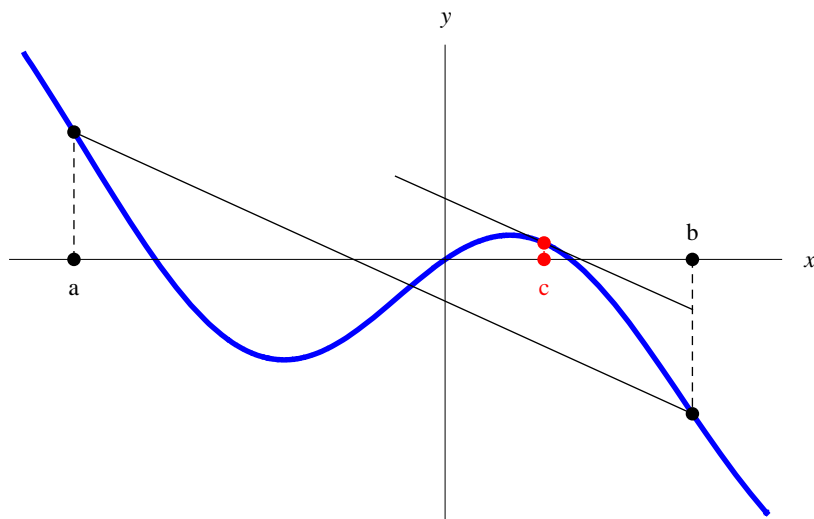
Figure 10: The Mean Value Theorem

Exercise 5.12

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b).

If $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing.

Exercise 5.13

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b).

If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.

A function $f : D \to \mathbb{R}$ is called INJECTIVE (or 1–1), if $x \neq y$ implies $f(x) \neq f(y)$ for all $x, y \in D$.

Exercise 5.14

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b).

If $f'(x) \neq 0$ for all $x \in (a, b)$, then f is injective.

Task 5.15

Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ such that $f'(x) \neq 0$ for all $x \in [a, b]$.

Then f is injective; its inverse f^{-1} is differentiable on $f([a, b])$. Moreover, setting $y = f(x)$, we have

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)}.$$

5.4   The Derivative and the Intermediate Value Property*

We say that a function $f : [a, b] \to \mathbb{R}$ has the INTERMEDIATE VALUE PROPERTY on $[a, b]$ if the following holds: Let $x_1, x_2 \in [a, b]$, and let

$$y \in (f(x_1), f(x_2)).$$

Then there is an $x \in (x_1, x_2)$ satisfying $f(x) = y$.

Recall that we saw earlier that every continuous function has the intermediate value property, see Task 4.17.

On the other hand, not every function with the intermediate value property is continuous:

Optional Task 5.3

Let $f : [-1, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0, \ x \in \mathbb{R} \\ 0, & \text{if } x = 0 \end{cases}$$

Show that f has the intermediate value property on the interval $[-1, 1]$. See Figure 5 on page 30.

The rest of this section will establish the surprising fact that derivatives have the intermediate value property, even though they are not necessarily continuous (see Task 5.3).

Optional Task 5.4

Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$.

If $f'(x) \neq 0$ for all $x \in (a, b)$, then either $f'(x) \geq 0$ for all $x \in [a, b]$ or $f'(x) \leq 0$ for all $x \in [a, b]$.

Optional Task 5.5

Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$. Then $f' : [a, b] \to \mathbb{R}$ has the intermediate value property on $[a, b]$.

5.5   A Continuous, Nowhere Differentiable Function*

This section follows the construction in [10]. Another example can be found in [12].

Recall that the LARGEST INTEGER FUNCTION $[x] : \mathbb{R} \to \mathbb{R}$ is defined as follows:

$$[x] = k, \text{ if } k \in \mathbb{Z} \text{ satisfies } k \leq x < k + 1.$$

For instance, $[4.5] = 4$ and $[-\pi] = -4$.

We start by defining a 1-periodic function $f_0 : \mathbb{R} \to \mathbb{R}$ as follows[10]:

$$f_0(x) = \begin{cases} x & \text{, if } 0 \leq x - [x] < \frac{1}{2} \\ 1 - x & \text{, if } \frac{1}{2} < x - [x] < 1 \end{cases}$$

See Figure 11 on the next page.

For $n \in \mathbb{N}$, we define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = 2^{-n} f_0(2^n x).$$

Figure 12 on page 51 depicts the function $f_2(x)$.

Finally we let $g_n : [0, 1] \to \mathbb{R}$ for $n \in \mathbb{N}$ be defined as

$$g_n(x) = \sum_{k=0}^{n} f_k(x),$$

and then set

$$g(x) = \lim_{n \to \infty} g_n(x)$$

---

[10]A function $f : \mathbb{R} \to \mathbb{R}$ is called p-periodic if $f(x + p) = f(x)$ for all $x \in \mathbb{R}$.
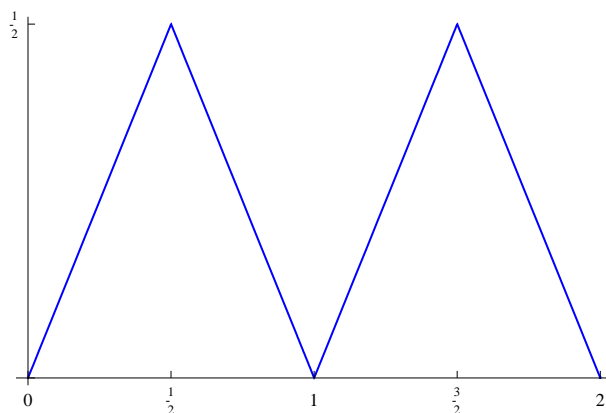
Figure 11: The function $f_0(x)$

for all $x \in [0, 1]$. Figure 13 on page 52 shows the function $g(x)$.

The function $g(x)$ is continuous on the interval $[0, 1]$, but fails to be differentiable at all points in the interval $(0, 1)$. To establish these properties we start with
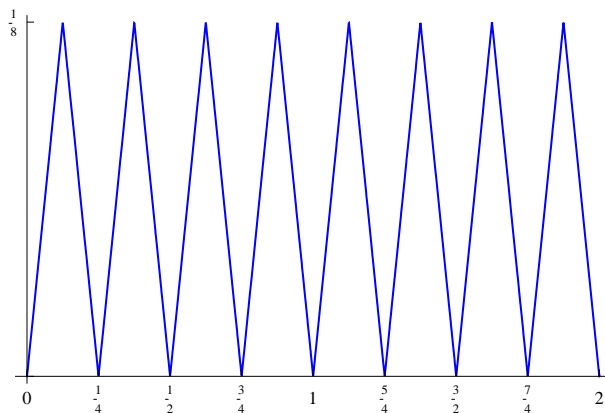

Optional Task 5.6

1. For $n \in \mathbb{N} \cup \{0\}$, the function $f_n(x)$ is continuous on $[0, 1]$ and 2^{-n}-periodic.

2. For $n \in \mathbb{N} \cup \{0\}$, the function $f_n(x)$ satisfies $0 \leq f_n(x) \leq 2^{-(n+1)}$ for all $x \in [0, 1]$.

3. Show that the estimate

$$|g_m(x) - g_n(x)| \leq 2^{-(1+\min\{m,n\})}$$

holds for all $x \in [0, 1]$ and all $m, n \in \mathbb{N} \cup \{0\}$.

4. Show that $g(x)$ is well-defined for all $x \in [0, 1]$.

5. The function $g(x)$ maps the interval $[0, 1]$ into itself.

Using the results above, show:

Figure 12: The function $f_2(x)$

Optional Task 5.7

The function $g(x)$ is continuous on $[0, 1]$.

We will now establish that the function $g(x)$ is nowhere differentiable. First we need the following result:

Optional Task 5.8

Let a function $f : [0, 1] \to \mathbb{R}$ be differentiable at the point $y \in (0, 1)$. Then

$$\lim \frac{f(z) - f(x)}{z - x} \text{ exists and equals } f'(y).$$

Here the limit is taken over all $x, z \in [0, 1]$ satisfying $x \leq y \leq z$ and $x \neq y$ such that $\max\{|y - x|, |z - y|\} \to 0$.

More precisely this means the following: For all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\left| \frac{f(z) - f(x)}{z - x} - f'(y) \right| < \varepsilon$$

for all $x, z \in [0, 1]$ satisfying $x \leq y \leq z$, $x \neq y$, $|y - x| < \delta$ and $|z - y| < \delta$.
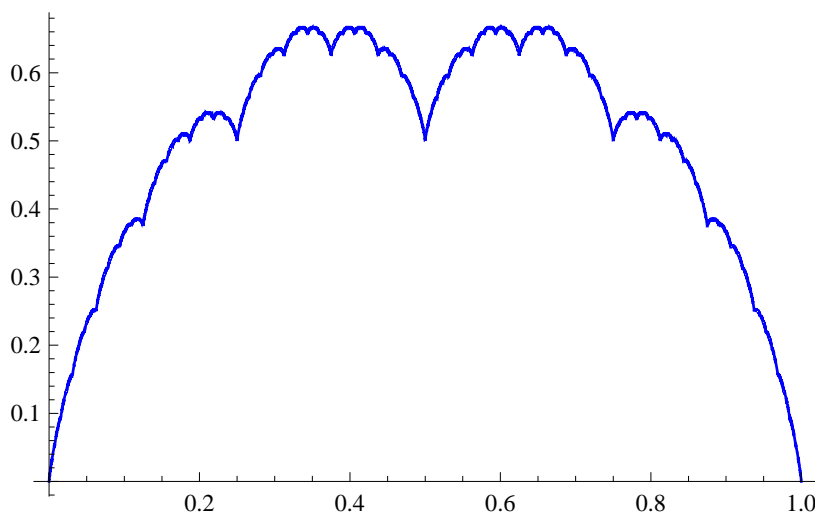
Figure 13: The function $g(x)$

The crucial step is the next task:

Optional Task 5.9
For all $y \in (0,1)$ there are four sequences (x_n), (x'_n), (z_n), and (z'_n) in $[0,1]$ with the following properties:

1. All four sequences converge to y.

2. $x_n \leq y \leq z_n$, $x_n \neq z_n$ for all $n \in \mathbb{N}$.

3. $x'_n \leq y \leq z'_n$, $x'_n \neq z'_n$ for all $n \in \mathbb{N}$.

4. $\left| \dfrac{g(z_n) - g(x_n)}{z_n - x_n} - \dfrac{g(z'_n) - g(x'_n)}{z'_n - x'_n} \right| \geq 1$ for all $n \in \mathbb{N}$.

The proof is somewhat technical. Let $p \in \mathbb{N}$ be such that $\dfrac{p}{2^n} \leq y < \dfrac{p+1}{2^n}$. Then choose x_n, z_n, x'_n and z'_n suitably from the set $\left\{ \dfrac{p}{2^n}, \dfrac{2p+1}{2^{n+1}}, \dfrac{p+1}{2^n} \right\}$. Figure 14 on page 54 shows a typical scenario (for $n = 11$ and $p = 172$).

Finally one can use the last two tasks to show:

Optional Task 5.10

The function $g : [0, 1] \to [0, 1]$ fails to be differentiable at all points in $(0, 1)$.

Since $g(x)$ is continuous on the interval $[0, 1]$, it has a maximum.

Optional Task 5.11

Show that the maximal value of $g(x)$ on the interval $[0, 1]$ is $\dfrac{2}{3}$.

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g(x) and g_{10}(x)

g(x) and g_{11}(x)
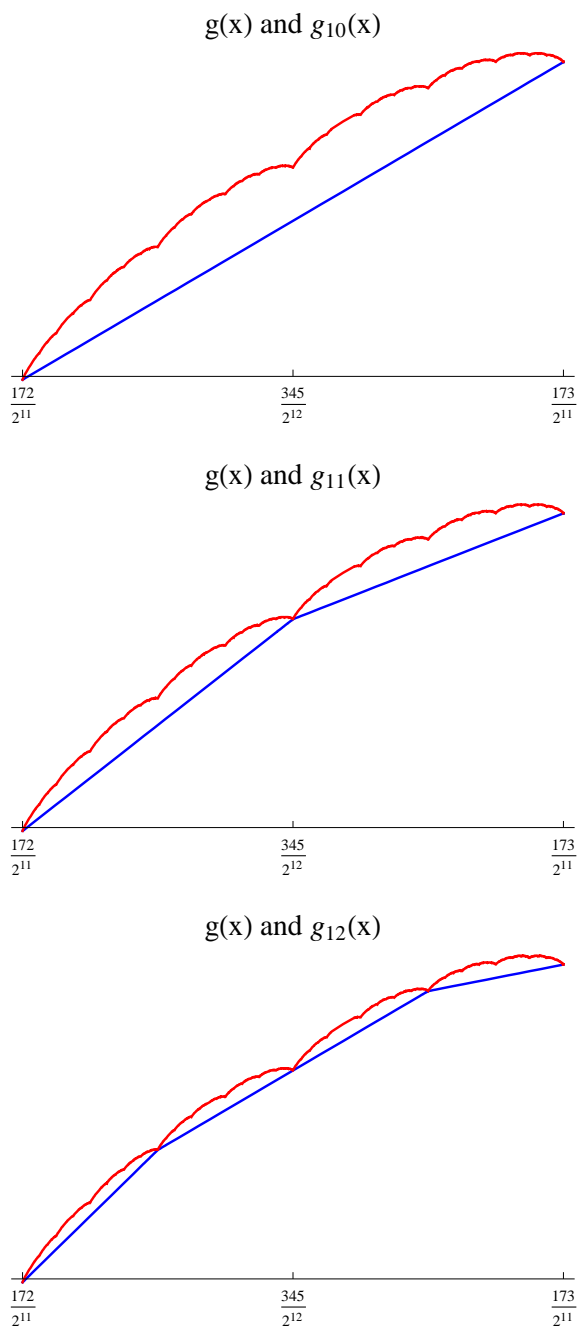
g(x) and g_{12}(x)

Figure 14: The pictures show the functions $g(x)$ and $g_{10}(x)$, $g(x)$ and $g_{11}(x)$, and $g(x)$ and $g_{12}(x)$, respectively.

6 The Integral

Throughout this chapter all functions are assumed to be bounded.

6.1 Definition and Examples

A finite set $P = \{x_0, x_1, x_2, \ldots, x_n\}$ of real numbers is called a PARTITION of the interval $[a, b]$, if

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$
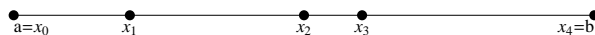
Figure 15: A 5-element partition of the interval $[a, b]$.

Let a function $f : [a, b] \to \mathbb{R}$ and a partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ of the interval $[a, b]$ be given. Let $i \in \{1, 2, 3, \ldots, n\}$. We define

$$m_i(f) := \inf\{f(x) \mid x \in [x_{i-1}, x_i]\},$$

and

$$M_i(f) := \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}.$$

The LOWER RIEMANN SUM $\mathcal{L}(f, P)$ of the function f with respect to the partition P is defined as

$$\mathcal{L}(f, P) := \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}).$$

See Figure 16 on the next page.

Analogously, the UPPER RIEMANN SUM $\mathcal{U}(f, P)$ is defined as

$$\mathcal{U}(f, P) := \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}).$$

See Figure 17 on page 57.
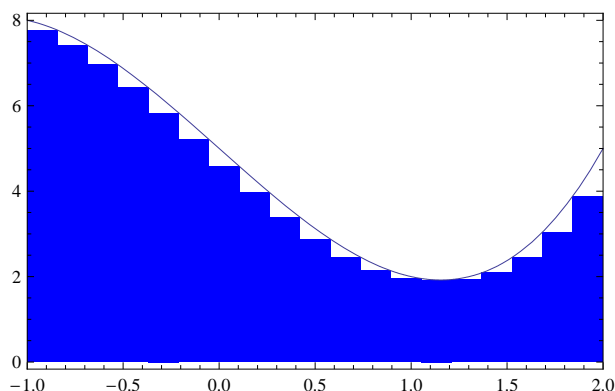
Figure 16: A lower Riemann sum with a partition of 20 equally spaced points.

The LOWER RIEMANN INTEGRAL of f on the interval $[a, b]$, denoted by $\mathcal{L}\int_a^b f(x)\,dx$, is defined as

$$\mathcal{L}\int_a^b f(x)\,dx := \sup\{\mathcal{L}(f, P) \mid P \text{ is a partition of } [a, b]\}.$$

Similarly, the UPPER RIEMANN INTEGRAL of f on the interval $[a, b]$ is defined as

$$\mathcal{U}\int_a^b f(x)\,dx := \inf\{\mathcal{U}(f, P) \mid P \text{ is a partition of } [a, b]\}.$$

Let P and Q be two partitions of the interval $[a, b]$. We say that the partition Q is FINER than the partition P if $P \subseteq Q$. In this situation, we also call P COARSER than Q.

Task 6.1

Let $f : [a, b] \to \mathbb{R}$ be a function, and P and Q be two partitions of the interval $[a, b]$. Assume that Q is finer than P. Then

$$\mathcal{L}(f, P) \leq \mathcal{L}(f, Q) \leq \mathcal{U}(f, Q) \leq \mathcal{U}(f, P).$$

Note that Task 6.1 implies that

$$\mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx.$$
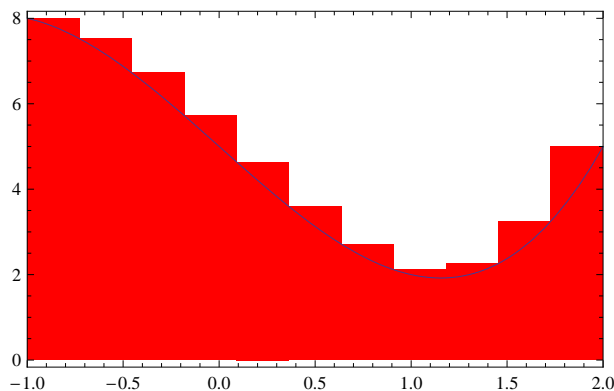
Figure 17: An upper Riemann sum with a partition of 12 equally spaced points.

We are finally in a position to define the concept of integrability! We will say that a function $f : [a, b] \to \mathbb{R}$ is RIEMANN INTEGRABLE on the interval $[a, b]$, if

$$\mathcal{L}\int_a^b f(x)\, dx = \mathcal{U}\int_a^b f(x)\, dx.$$

Their common value is then called the RIEMANN INTEGRAL of f on the interval $[a, b]$ and denoted by

$$\int_a^b f(x)\, dx.$$

Exercise 6.2

Use the definition above to compute $\mathcal{L}\int_0^1 x\, dx$ and $\mathcal{U}\int_0^1 x\, dx$. Is the function Riemann integrable on $[0, 1]$?

Exercise 6.3

Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Use the definitions above to compute $\mathcal{L}\int_0^1 f(x)\, dx$ and $\mathcal{U}\int_0^1 f(x)\, dx$. Is the function Riemann integrable on $[0, 1]$?

Task 6.4

A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if for every $\varepsilon > 0$ there is a partition P of $[a, b]$ such that

$$\mathcal{U}(f, P) - \mathcal{L}(f, P) < \varepsilon.$$

The following "Lemma" is quite technical; it prepares for the proof of the subsequent result. Note that given two partitions P and Q, the partition $P \cup Q$ is finer than both P and Q.

Given a partition P, we define its MESH WIDTH $\mu(P)$ as

$$\mu(P) := \max\{x_i - x_{i-1} \mid i = 1, 2, \ldots, n\}.$$

Task 6.5

Let $f : [a, b] \to \mathbb{R}$ be a bounded function with $|f(x)| \le M$ for all $x \in [a, b]$. Let $\varepsilon > 0$ be given, and let P_0 be a partition of $[a, b]$ with $n + 1$ elements. Then there is a $\delta > 0$ (depending on M, n and ε) such that for any partition P of $[a, b]$ with mesh width $\mu(P) < \delta$

$$\mathcal{U}(P) < \mathcal{U}(P \cup P_0) + \varepsilon \qquad \text{and} \qquad \mathcal{L}(P) > \mathcal{L}(P \cup P_0) - \varepsilon.$$

Task 6.6

A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for **all** partitions P of $[a, b]$ with mesh width $\mu(P) < \delta$

$$\mathcal{U}(f, P) - \mathcal{L}(f, P) < \varepsilon.$$

Two important classes of functions are Riemann-integrable—continuous functions and monotone functions:

Task 6.7

If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then f is Riemann integrable on $[a, b]$.

Task 6.8

If $f : [a, b] \to \mathbb{R}$ is increasing on $[a, b]$, then f is Riemann integrable on $[a, b]$.

An analogous result holds for decreasing functions, of course.

6.2   Arithmetic of Integrals

Exercise 6.9

Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$. Then for all $\lambda \in \mathbb{R}$, the function λf is also Riemann integrable on $[a, b]$, and

$$\int_a^b \lambda f(x)\, dx = \lambda \int_a^b f(x)\, dx.$$

Exercise 6.10

Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$. Then $f + g$ is also Riemann integrable on $[a, b]$, and

$$\int_a^b f(x) + g(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$$

Task 6.11

Let $f : [a, c] \to \mathbb{R}$ be a function and $a < b < c$. Then f is Riemann integrable on $[a, c]$ if and only if f is Riemann integrable on both $[a, b]$ and $[b, c]$. In this case

$$\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx.$$

Exercise 6.12

Suppose the function $f : [a, b] \to \mathbb{R}$ is bounded above by $M \in \mathbb{R}$: $f(x) \leq M$ for all $x \in [a, b]$. Then

$$\int_a^b f(x)\, dx \leq M \cdot (b - a).$$

6.3   The Fundamental Theorem of Calculus

Task 6.13

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Then there is a number $c \in [a, b]$ such that

$$f(c) = \frac{1}{b - a} \int_a^b f(x)\, dx.$$

Task 6.14

Let $f : [a, b] \to \mathbb{R}$ be bounded and Riemann integrable on $[a, b]$. Let

$$F(x) = \int_a^x f(\tau)\, d\tau.$$

Then $F : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$.

Does your proof of the result above actually yield a stronger result?

Task 6.15

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Let

$$F(x) = \int_a^x f(\tau)\, d\tau.$$

Then $F : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$, and

$$F'(x) = f(x).$$

The next result is the **Fundamental Theorem of Calculus**.

Task 6.16

Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$, and Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$, and suppose $F : [a, b] \to \mathbb{R}$ is an "anti-derivative" of $f(x)$, i.e., F satisfies:

1. F is continuous on $[a, b]$ and differentiable on (a, b),

2. $F'(x) = f(x)$ for all $x \in [a, b]$.

Then

$$\int_a^b f(\tau) \, d\tau = F(b) - F(a).$$

A   Cardinality*

Recall that a function $\varphi : A \to B$ is called INJECTIVE, if $x \neq x'$ implies $\varphi(x) \neq \varphi(x)$ for all elements $x, x' \in A$. A function $\varphi : A \to B$ is called SURJECTIVE, if for all elements $y \in B$ there is an element $x \in A$ with $\varphi(x) = y$. A BIJECTION is a function that is both injective and surjective.

We say that two sets A and B have the same CARDINALITY if there is a bijection $\varphi : A \to B$. If A and B have the same cardinality we write $A \sim B$.

Given a set G, its POWER SET $\mathcal{P}(G)$ is the set of all subsets of G:

$$\mathcal{P}(G) := \{A \mid A \subseteq G\}$$

For instance, $\mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.

Task A.1

Let G be some set. Then "$\sim$" defines an equivalence relation on the power set of G, *i.e.,* the following properties hold:

1. $A \sim A$ for all $A \in \mathcal{P}(G)$

2. If $A \sim B$, then $B \sim A$ for all $A, B \in \mathcal{P}(G)$

3. If $A \sim B$ and $B \sim C$, then $A \sim C$ holds for all $A, B, C \in \mathcal{P}(G)$

Let $n \in \mathbb{N}$ and A be a set. If $A \sim \{1, 2, 3, \ldots, n\}$, we say A has cardinality n. Sets with cardinality n for some $n \in \mathbb{N}$ are called FINITE. The empty set $\emptyset$ is said to have cardinality 0 and is also considered to be a finite set.

A set A, for which $A \sim \mathbb{N}$ is called COUNTABLE, or said to have COUNTABLE CARDINAL-ITY.

The next result is usually attributed to Galileo Galilei (1564–1642):

Task A.2

Let $2\mathbb{N}$ be the set of even natural numbers. Show: $\mathbb{N} \sim 2\mathbb{N}$.

The set of natural numbers has the same cardinality as the set of all integers:

Task A.3

$\mathbb{N} \sim \mathbb{Z}$.

The set of rational numbers is also countable; this result like all the following results in this section were discovered by Georg Cantor (1845–1918).

Task A.4

$\mathbb{N} \sim \mathbb{Q}$.

You may want to prove first that the set of positive rational numbers is countable.

Are there uncountable sets? The answer to this question is a loud yes! The next result establishes that $\mathcal{P}(\mathbb{N})$ is an example of an infinite set that is not countable.

This result, known as **Cantor's Theorem**, is considered to be one of the most beautiful results in all of mathematics. It generalizes the well-known fact that for finite sets G of cardinality n, the power set $\mathcal{P}(G)$ has cardinality 2^n.

Task A.5

Let A be any set. Then $\mathcal{P}(A)$ does not have the same cardinality as A.

How does one go about proving something like this? A direct proof seems to be impossible—the only hope is a proof by contradiction...

Cantor's Theorem shows that there is at least a countable number of distinct uncountable cardinalities.

The rest of this section is devoted to a proof of the fact that $\mathbb{R} \sim \mathcal{P}(\mathbb{N})$, thus establishing that the set of real numbers is uncountable.

Task A.6

$\mathbb{R} \sim (0, 1)$.

Open and closed intervals have the same cardinality:

Task A.7
$(0, 1) \sim [0, 1]$.

It is left to establish that $[0, 1] \sim \mathcal{P}(\mathbb{N})$. To see this we will write elements $x \in [0, 1]$ in their BINARY EXPANSION.

For instance $\dfrac{5}{8} = \dfrac{1}{2^1} + \dfrac{0}{2^2} + \dfrac{1}{2^3}$ and $1 = \dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \dfrac{1}{2^4} + \dots$

Task A.8
Show that $\quad \dfrac{1}{3} = \dfrac{1}{2^2} + \dfrac{1}{2^4} + \dfrac{1}{2^6} + \dfrac{1}{2^8} + \dots$

Task A.9
Show that every $x \in [0, 1]$ can be written as

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{2^n}$$

with ε_n either 0 or 1 for all $n \in \mathbb{N}$.

The next two tasks address a minor technical problem: the binary expansion of a real number is not always unique; some numbers in $[0, 1]$ have two different binary expansions.

Task A.10
Let B be the set of those real numbers in $[0, 1]$ with two (or more?) distinct binary expansions.

1. Find an element in B.

2. Classify all real numbers in B.

3. Show that the set B is countable (and infinite).

Task A.11
Suppose $A \sim \mathcal{P}(\mathbb{N})$, and let $B \subseteq A$ with $B \sim \mathbb{N}$. Then

$$(A \setminus B) \sim \mathcal{P}(\mathbb{N}).$$

Here is the result we are after:

Task A.12
$[0, 1] \sim \mathcal{P}(\mathbb{N})$.

Is it true that every set A of real numbers containing $\mathbb{N}$ either has the cardinality of $\mathbb{N}$ or the cardinality of $\mathbb{R}$? This question is known as the CONTINUUM HYPOTHESIS:

$$\text{Does } \mathbb{N} \subset A \subset \mathbb{R} \text{ imply that either } A \sim \mathbb{N} \text{ or } A \sim \mathbb{R}?$$

The Continuum Hypothesis cannot be answered. Within the usual axiomatic system of set theory, no contradictions arise from either a positive or a negative answer to the Continuum Hypothesis. This deep result that the Continuum Hypothesis is independent of the axioms of set theory was proved by Paul Cohen (1934–) in 1963, completing work started by Kurt Gödel (1906–1978).

B   The Cantor Set*

A non-empty set is called PERFECT if it equals its set of accumulation points. $\mathbb{R}$ itself and closed intervals are examples of perfect sets.

Task B.1

Every perfect set is uncountable.

Note that this result provides a second proof of the fact that $\mathbb{R}$ is uncountable, without giving the additional information that $\mathbb{R} \sim \mathcal{P}(\mathbb{N})$.

We now construct a more interesting perfect set: Let $C_0 = [0, 1]$. Let's remove the middle third of C_0:

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

Next we remove the two middle thirds of C_1:

$$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

Continue in this fashion: in each step remove the middle thirds of the intervals from the previous step. See Figure 18 on the next page.

The CANTOR SET C is defined as the intersection of all these sets:

$$C = \bigcap_{n=0}^{\infty} C_n.$$

Task B.2

Show that C is perfect.

Task B.3

Show that C does not contain an open interval.
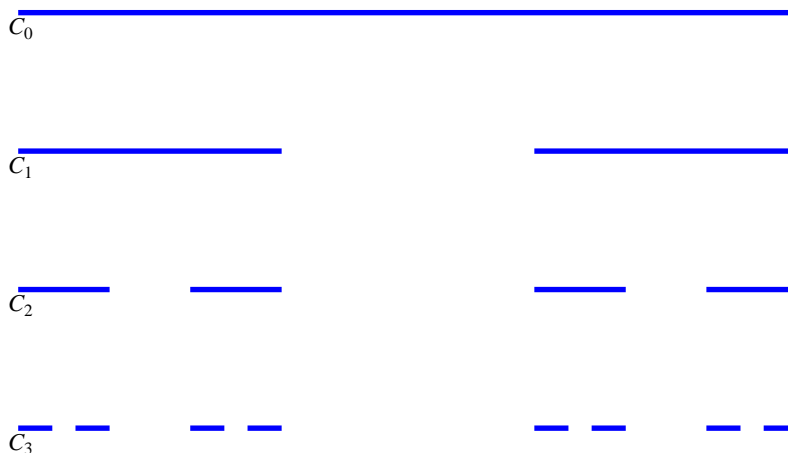
Figure 18: The first steps in building the Cantor set.

What is the "length" of the Cantor set? Clearly C_0 has length 1 (being an interval of length 1).

Similarly, C_1 has length $\frac{2}{3}$, C_2 has length $\frac{2^2}{3^2}$, etc. Since C is contained in all C_n, it must have "length" 0.

The notion of "length" will be made precise in a course on measure theory. You will learn in such a course that C is indeed a set of measure 0.

Task B.4

The Cantor set consists of all real numbers in $[0, 1]$, which have a base-3 expansion with digits 0 and 2:

$$C = \left\{ \sum_{n=1}^{\infty} \frac{\varepsilon_n}{3^n} \mid \varepsilon_n \in \{0, 2\} \text{ for all } n \in \mathbb{N} \right\}.$$

Task B.4 provides an alternative way to show that C is uncountable, and indeed that

C has the same cardinality as $\mathbb{R}$. Let $\varphi : [0, 1] \to C$, be defined by

$$\varphi\left(\sum_{n=1}^{\infty} \frac{\varepsilon_n}{2^n}\right) = \sum_{n=1}^{\infty} \frac{2\varepsilon_n}{3^n}.$$

Here, every real number $x \in [0, 1]$ is written in its binary expansion as

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{2^n}$$

with $\varepsilon_n \in \{0, 1\}$ for all $n \in \mathbb{N}$. The map "essentially" is a bijection; there is a small problem again with those real numbers that have two distinct binary expansions.

How to Solve It

George Pólya (1887–1985) described the experience of problem solving in his book *How to Solve It* [18], p. v:

> *A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive facilities, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery.*

As part of his work on problem solving, Pólya developed a four-step problem-solving process similar to the following:

Understanding the Problem

1. Can you state the problem in your own words?
2. What are you trying to find or do?
3. What are the unknowns?
4. What information do you obtain from the problem?
5. What information, if any, is missing or not needed?

Devising a Plan The following list of strategies, although not exhaustive, is very useful:

1. Look for a pattern.
2. Examine related problems and determine if the same technique can be applied.
3. Examine a simpler or special case of the problem to gain insight into the solution of the original problem.
4. Make a table.
5. Make a diagram.
6. Write an equation.
7. Use a guess and check.
8. Work backward.
9. Identify a subgoal.

Carrying out the Plan

1. Implement the strategy in Step 2 and perform any necessary actions or computations.

2. Check each step of the plan as you proceed. This may be intuitive checking or a formal proof of each step.

3. Keep an accurate record of your work.

Looking Back

1. Check the results in the original problem. In some cases, this will require a proof.

2. Interpret the solution in terms of the original problem. Does your answer make sense? Is it reasonable?

3. Determine whether there is another method of finding the solution.

4. If possible, determine other related or more general problems for which the techniques will work.

These and other general problem-solving strategies, or rules of thumb for successful problem solving, are called heuristics.

How to Check Your Written Proofs

The objective of a proof is to show an informed reader (e.g. a fellow mathematician), why the statement under consideration is correct. Because of its communicative nature, a proof has to satisfy the same standards as other technical writing: It has to be correct (your main concern!), express its thoughts clearly, explain its ideas in the easiest way possible, be coherent, legible and aesthetically pleasing.

Alternating between "proofreading" your proof line-by-line and considering your "product" as a whole is one way to achieve these goals.

Line-by-line Analysis. While you are carefully checking each line and each little step of your proof, you should watch out for the following:

- Is this step correct?

- Are there counterexamples?

- Are all symbols defined or explained, the first time they show up in the proof?

- Do I need all the symbols and steps I use?

- Is the spelling correct?

- Can the wording be improved upon?

- Is there a more elegant way of explaining the argument?

Even making minor changes during a line-by-line analysis usually requires to start the analysis all over again. If you make more than minor changes, you have to rewrite your proof completely. Proofs you have seen in books have probably been rewritten by the author more than a dozen times.

Global Analysis. During a global analysis you consider your proof as a whole:

- Does my proof "really" show what I am supposed to show?

- Did I forget to prove any of the statements?

- Are all parts of my proof really necessary?

- Do I use all the hypotheses?

- Do I give all necessary references and acknowledgments?

- Does one need all the hypotheses? Does my proof suggest generalizations?

- Does my "final product" look good?

Acknowledgments and References. If you work together with classmates and jointly obtain a solution to a problem, proper credit must be given, e.g., ... *jointly obtained by J. Doe and myself; we thank J. Doe for her helpful advice...*

There is a fine line between academic cooperation and collusion. To avoid the latter, it is recommended when working in a group, that the participants independently compose their own final version of the proof. Copying a solution from a classmate or using other sources without references constitutes an act of academic dishonesty.

Greek Alphabet

α	alpha	β	beta	γ	gamma	δ	delta
ϵ, ε	epsilon	ζ	zeta	η	eta	θ, ϑ	theta
ι	iota	κ	kappa	λ	lambda	μ	mu
ν	nu	ξ	xi	o	omikron	π	pi
ρ, ϱ	rho	σ	sigma	τ	tau	υ	upsilon
ϕ, φ	phi	χ	chi	ψ	psi	ω	omega
Γ	Gamma	Δ	Delta	Θ	Theta	Λ	Lambda
Ξ	Xi	Π	Pi	Σ	Sigma	Υ	Upsilon
Φ	Phi	Ψ	Psi	Ω	Omega		

Figure 19: Some letters of the Greek Alphabet and their pronunciation

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