Math 3335 **Solutions for Test 2** **Fall 2016**

Problem 1 (10 points) Consider $f : \mathbb{R}^3 \to \mathbb{R}$, given by

$$f(x, y, z) = 2x^2z^2 - 3y^2z.$$

1. At the point $(-1, 2, 1)$, find the direction in which f **increases** most rapidly.

   The function increases most rapidly in the direction of the gradient.

   $$\nabla f(x, y, z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (4xz^2, -6yz, 4x^2z - 3y^2).$$

   Consequently $\nabla f(-1, 2, 1) = (-4, -12, -8)$.

2. Compute the directional derivative in this direction.

   If $\vec{\boldsymbol{v}}$ is a unit vector, then the directional derivative of f in the direction of $\vec{\boldsymbol{v}}$ is given by $\nabla f \cdot \vec{\boldsymbol{v}}$. In our case $\vec{\boldsymbol{v}} = \dfrac{\nabla f}{\|\nabla f\|}$, and thus the desired directional derivative is

   $$\nabla f(-1, 2, 1) \cdot \frac{\nabla f(-1, 2, 1)}{\|\nabla f(-1, 2, 1)\|} = \|\nabla f(-1, 2, 1)\| = \sqrt{224} = 4\sqrt{14}.$$

Problem 2 (15 points) Let $\vec{\boldsymbol{f}} : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$\vec{\boldsymbol{f}}(x, y) = (2xy - y\sin(xy) + \cos(x))\vec{\boldsymbol{\imath}} + (x^2 - x\sin(xy) - 3y^2)\vec{\boldsymbol{\jmath}}.$$

1. Find $F : \mathbb{R}^2 \to \mathbb{R}$ such that $\nabla F = \vec{\boldsymbol{f}}$.

   $\dfrac{\partial F}{\partial x} = 2xy - y\sin(xy) + \cos(x)$, so

   $$F(x, y) = \int (2xy - y\sin(xy) + \cos(x))dx = x^2y + \cos(xy) + \sin(x) + g(y).$$

   To determine the function $g(y)$, we differentiate the expression above with respect to y, and compare to the second component of $\vec{\boldsymbol{f}}(x, y)$:

   $$\frac{\partial F}{\partial y} = x^2 - x\sin(xy) + g'(y).$$

   Thus $g'(y) = -3y^2$; so $g(y) = -y^3$ and

   $$F(x, y) = x^2y + \cos(xy) + \sin(x) - y^3.$$

2. Let C be the curve parametrized by $\vec{\boldsymbol{r}}(t) = (t, 2t)$ for $t \in [0, 1]$. Compute

   $$\int_C \vec{\boldsymbol{f}} \cdot d\vec{\boldsymbol{r}}.$$

   Using the FTC for line integrals, we obtain

   $$\int_C \vec{\boldsymbol{f}} \cdot d\vec{\boldsymbol{r}} = F(\vec{\boldsymbol{r}}(1)) - F(\vec{\boldsymbol{r}}(0)) = F(1, 2) - F(0, 0) = -7 + \sin(1) + \cos(2).$$

Problem 3 (15 points) Let

$$\vec{\boldsymbol{f}}(t) = \left(\frac{\cos t}{\sqrt{1+t^2}}, \frac{\sin t}{\sqrt{1+t^2}}, \frac{-t}{\sqrt{1+t^2}} \right).$$

1. Compute $\|\vec{\boldsymbol{f}}(t)\|$.

$$\|\vec{\boldsymbol{f}}(t)\| = \sqrt{\frac{\cos^2 t + \sin^2 t + t^2}{1+t^2}} = 1.$$

2. Show that $\vec{\boldsymbol{f}}'(t) \cdot \vec{\boldsymbol{f}}(t) = 0$ for all t.

$\vec{\boldsymbol{f}}'(t) = \frac{1}{(1+t^2)^{1/2}}(-\sin t, \cos t, -1) - \frac{t}{(1+t^2)^{3/2}}(\cos t, \sin t, -t)$, using the quotient rule three times (with the same denominator). Consequently,

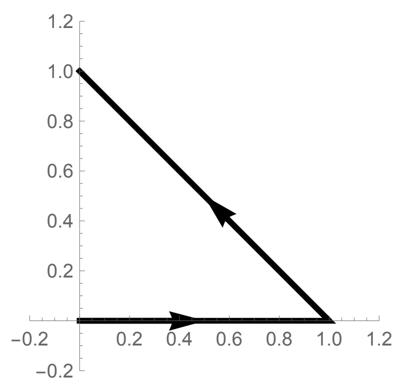
$$\begin{aligned} \vec{\boldsymbol{f}}'(t) \cdot \vec{\boldsymbol{f}}(t) &= \frac{1}{1+t^2}(-\sin t \cos t + \sin t \cos t + t) - \frac{t}{(1+t^2)^2}(\cos^2 t + \sin^2 + t^2) \\ &= \frac{t}{1+t^2} - \frac{t}{(1+t^2)^2}(1+t^2) = 0. \end{aligned}$$

Problem 4 (15 points) Compute the arclength of $\vec{\boldsymbol{f}}(t) = (\cos 3t, \sin 3t, 2t^{3/2})$ on the interval $[0, 1]$.

$$\int_0^1 \|\vec{\boldsymbol{f}}'(t)\|\, dt = \int_0^1 \sqrt{9\sin^2 t + 9\cos^2 t + 9t}\, dt = 3\int_0^1 \sqrt{1+t}\, dt = 4\sqrt{2} - 2.$$

In the last step one can use a substitution $u = 1 + t$.

Problem 5 (15 points) Compute the line integral for $\vec{\boldsymbol{f}}(x, y) = (x^2 - e^x, x + e^{2y})$ over the polygonal path from $(0, 0)$ to $(1, 0)$ to $(0, 1)$.



We can parametrize the horizontal portion of the curve by $\vec{\boldsymbol{r}}(t) = (t, 0)$, for $t \in [0, 1]$, and the diagonal portion of the curve by $\vec{\boldsymbol{r}}(t) = (1 - t, t)$, also with $t \in [0, 1]$.

Thus

$$\begin{aligned}
\int_C \vec{\boldsymbol{f}} \cdot d\vec{\boldsymbol{r}} &= \int_0^1 (t^2 - e^t, t+1) \cdot (1,0)\, dt \\
&\qquad + \int_0^1 ((1-t)^2 + e^{1-t}, (1-t) + e^{2t}) \cdot (-1,1)\, dt \\
&= \int_0^1 t^2 - e^t - (1-t)^2 - e^{1-t} + (1-t) + e^{2t}\, dt \\
&= \frac{e^2}{2}
\end{aligned}$$

Problem 6 (15 points) Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be two continuously differentiable functions. (∇ denotes the gradient operator.)

1. Show: $\nabla(fg) = f\ \nabla g + g\ \nabla f$.

$$\begin{aligned}
\nabla(fg) &= \left(\frac{\partial(fg)}{\partial x}, \frac{\partial(fg)}{\partial y}\right) = \left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}, f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right) \\
&= f\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right) + g\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = f\ \nabla g + g\ \nabla f.
\end{aligned}$$

2. Let C be a **closed** curve. Explain why

$$\oint_C (f\ \nabla g) \cdot d\vec{\boldsymbol{r}} = -\oint_C (g\ \nabla f) \cdot d\vec{\boldsymbol{r}}.$$

This follows from Part 1 and the FTC for line integrals over closed curves:

$$\oint_C (f\ \nabla g) \cdot d\vec{\boldsymbol{r}} + \oint_C (g\ \nabla f) \cdot d\vec{\boldsymbol{r}} = \oint_C \nabla(fg) \cdot d\vec{\boldsymbol{r}} = 0.$$

Problem 7 (15 points) Let $R = \{(x,y) \mid x^2 + y^2 \leq 1 \text{ and } y \geq 0\}$ be the semi-disk of radius 1 centered at $(0,0)$, and let $\vec{\boldsymbol{F}}(x,y) = (2, x^2)$.

1. Compute the left side of Green's Theorem.

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \int_{-1}^1 \left(\int_0^{\sqrt{1-x^2}} 2x\, dy\right) dx = \int_{-1}^1 2x\sqrt{1-x^2}\, dx = 0.$$

For the last step a substitution $u = 1 - x^2$ is helpful.

2. Compute the right side of Green's Theorem.

Let C_1 be the circular portion of the boundary of R, and let C_2 be the straight portion at the bottom of R. C_1 can be parametrized by $\vec{\boldsymbol{r}}(t) = (\cos t, \sin t)$ for $t \in [0, \pi]$, and a parametrization of C_2 is $\vec{\boldsymbol{r}}(t) = (t, 0)$, with $t \in [-1, 1]$. It is then straightforward to check that the line integral over C_1 equals -4, while the second line integral computes to $+4$.