Chapter Three

Elementary Functions

3.1. Introduction. Complex functions are, of course, quite easy to come by—they are simply ordered pairs of real-valued functions of two variables. We have, however, already seen enough to realize that it is those complex functions that are differentiable that are the most interesting. It was important in our invention of the complex numbers that these new numbers in some sense included the old real numbers—in other words, we extended the reals. We shall find it most useful and profitable to do a similar thing with many of the familiar real functions. That is, we seek complex functions such that when restricted to the reals are familiar real functions. As we have seen, the extension of polynomials and rational functions to complex functions is easy; we simply change x's to z's. Thus, for instance, the function f defined by

$$f(z) = \frac{z^2 + z + 1}{z + 1}$$

has a derivative at each point of its domain, and for $z = x + 0i$, becomes a familiar real rational function

$$f(x) = \frac{x^2 + x + 1}{x + 1}.$$

What happens with the trigonometric functions, exponentials, logarithms, *etc*., is not so obvious. Let us begin.

3.2. **The exponential function**. Let the so-called exponential function exp be defined by

$$\exp(z) = e^x(\cos y + i\sin y),$$

where, as usual, $z = x + iy$. From the Cauchy-Riemann equations, we see at once that this function has a derivative every where—it is an entire function. Moreover,

$$\frac{d}{dz}\exp(z) = \exp(z).$$

Note next that if $z = x + iy$ and $w = u + iv$, then

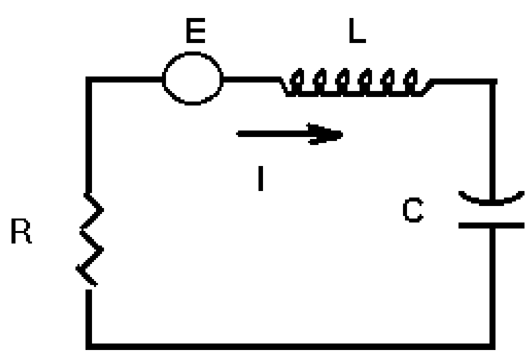
$$\begin{aligned}\exp(z+w) &= e^{x+u}[\cos(y+v)+i\sin(y+v)] \\ &= e^{x}e^{u}[\cos y\cos v - \sin y\sin v + i(\sin y\cos v + \cos y\sin v)] \\ &= e^{x}e^{u}(\cos y + i\sin y)(\cos v + i\sin v) \\ &= \exp(z)\exp(w).\end{aligned}$$

We thus use the quite reasonable notation $e^z = \exp(z)$ and observe that we have extended the real exponential e^x to the complex numbers.

Example

Recall from elementary circuit analysis that the relation between the voltage drop V and the current flow I through a resistor is $V = RI$, where R is the resistance. For an inductor, the relation is $V = L\frac{dI}{dt}$, where L is the inductance; and for a capacitor, $C\frac{dV}{dt} = I$, where C is the capacitance. (The variable t is, of course, time.) Note that if V is sinusoidal with a frequency ω, then so also is I. Suppose then that $V = A\sin(\omega t + \varphi)$. We can write this as $V = \mathrm{Im}(Ae^{i\varphi}e^{i\omega t}) = \mathrm{Im}(Be^{i\omega t})$, where B is complex. We know the current I will have this same form: $I = \mathrm{Im}(Ce^{i\omega t})$. The relations between the voltage and the current are linear, and so we can consider complex voltages and currents and use the fact that $e^{i\omega t} = \cos\omega t + i\sin\omega t$. We thus assume a more or less fictional complex voltage V, the imaginary part of which is the actual voltage, and then the actual current will be the imaginary part of the resulting complex current.

What makes this a good idea is the fact that differentiation with respect to time t becomes simply multiplication by $i\omega$: $\frac{d}{dt}Ae^{i\omega t} = i\omega Ae^{i\omega t}$. If $I = be^{i\omega t}$, the above relations between current and voltage become $V = i\omega LI$ for an inductor, and $i\omega VC = I$, or $V = \frac{I}{i\omega C}$ for a capacitor. Calculus is thereby turned into algebra. To illustrate, suppose we have a simple RLC circuit with a voltage source $V = a\sin\omega t$. We let $E = ae^{iwt}$.



Then the fact that the voltage drop around a closed circuit must be zero (one of Kirchoff's celebrated laws) looks like

$$i\omega LI + \frac{I}{i\omega C} + RI = ae^{i\omega t}, \text{or}$$
$$i\omega Lb + \frac{b}{i\omega C} + Rb = a$$

Thus,

$$b = \frac{a}{R + i\left(\omega L - \frac{1}{\omega C}\right)}.$$

In polar form,

$$b = \frac{a}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} e^{i\varphi},$$

where

$$\tan\varphi = \frac{\omega L - \frac{1}{\omega C}}{R}. \quad (R \neq 0)$$

Hence,

$$I = \mathrm{Im}(be^{i\omega t}) = \mathrm{Im}\left(\frac{a}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} e^{i(\omega t + \varphi)} \right)$$
$$= \frac{a}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin(\omega t + \varphi)$$

This result is well-known to all, but I hope you are convinced that this algebraic approach afforded us by the use of complex numbers is far easier than solving the differential equation. You should note that this method yields the steady state solution—the transient solution is not necessarily sinusoidal.

Exercises

1. Show that $\exp(z + 2\pi i) = \exp(z)$.

2. Show that $\frac{\exp(z)}{\exp(w)} = \exp(z - w)$.

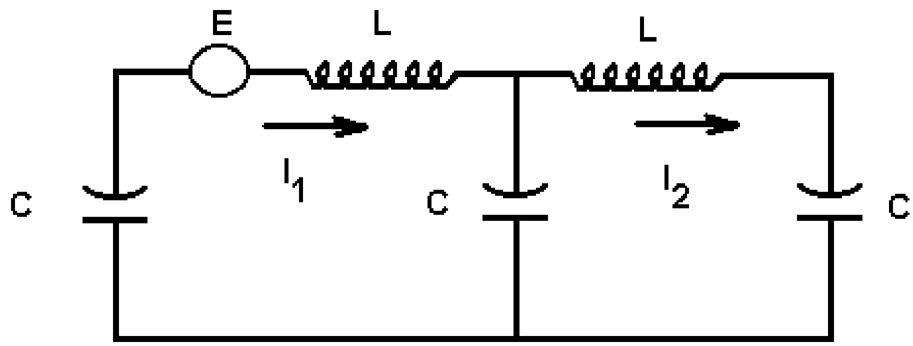
3. Show that $|\exp(z)| = e^x$, and $\arg(\exp(z)) = y + 2k\pi$ for any $\arg(\exp(z))$ and some integer k.

4. Find all z such that $\exp(z) = -1$, or explain why there are none.

5. Find all z such that $\exp(z) = 1 + i$, or explain why there are none.

6. For what complex numbers w does the equation $\exp(z) = w$ have solutions? Explain.

7. Find the indicated mesh currents in the network:



3.3 Trigonometric functions. Define the functions cosine and sine as follows:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2},$$
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

where we are using $e^z = \exp(z)$.

First, let's verify that these are honest-to-goodness extensions of the familiar real functions, cosine and sine–otherwise we have chosen very bad names for these complex functions. So, suppose $z = x + 0i = x$. Then,

$$e^{ix} = \cos x + i\sin x, \text{ and}$$
$$e^{-ix} = \cos x - i\sin x.$$

Thus,

$$\cos x = \frac{e^{ix} + e^{-ix}}{2},$$
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i},$$

and everything is just fine.

Next, observe that the sine and cosine functions are entire–they are simply linear combinations of the entire functions e^{iz} and e^{-iz}. Moreover, we see that

$$\frac{d}{dz}\sin z = \cos z, \text{ and } \frac{d}{dz}\cos z = -\sin z,$$

just as we would hope.

It may not have been clear to you back in elementary calculus what the so-called hyperbolic sine and cosine functions had to do with the ordinary sine and cosine functions. Now perhaps it will be evident. Recall that for real t,

$$\sinh t = \frac{e^t - e^{-t}}{2}, \text{ and } \cosh t = \frac{e^t + e^{-t}}{2}.$$

Thus,

$$\sin(it) = \frac{e^{i(it)} - e^{-i(it)}}{2i} = i\frac{e^t - e^{-t}}{2} = i\sinh t.$$

Similarly,

$$\cos(it) = \cosh t.$$

How nice!

Most of the identities you learned in the 3^{rd} grade for the real sine and cosine functions are also valid in the general complex case. Let's look at some.

$$\begin{aligned}\sin^2 z + \cos^2 z &= \frac{1}{4}[-(e^{iz} - e^{-iz})^2 + (e^{iz} + e^{-iz})^2] \\ &= \frac{1}{4}[-e^{2iz} + 2e^{iz}e^{-iz} - e^{-2iz} + e^{2iz} + 2e^{iz}e^{-iz} + e^{-2iz}] \\ &= \frac{1}{4}(2+2) = 1\end{aligned}$$

It is also relative straight-forward and easy to show that:

$$\sin(z \pm w) = \sin z \cos w \pm \cos z \sin w, \text{ and}$$
$$\cos(z \pm w) = \cos z \cos w \mp \sin z \sin w$$

Other familiar ones follow from these in the usual elementary school trigonometry fashion.

Let's find the real and imaginary parts of these functions:

$$\begin{aligned}\sin z &= \sin(x+iy) = \sin x \cos(iy) + \cos x \sin(iy)\\ &= \sin x \cosh y + i \cos x \sinh y.\end{aligned}$$

In the same way, we get $\cos z = \cos x \cosh y - i \sin x \sinh y$.

Exercises

8. Show that for all z,
a) $\sin(z + 2\pi) = \sin z$; b) $\cos(z + 2\pi) = \cos z$; c) $\sin\left(z + \frac{\pi}{2}\right) = \cos z$.

9. Show that $|\sin z|^2 = \sin^2 x + \sinh^2 y$ and $|\cos z|^2 = \cos^2 x + \sinh^2 y$.

10. Find all z such that $\sin z = 0$.

11. Find all z such that $\cos z = 2$, or explain why there are none.

3.4. Logarithms and complex exponents. In the case of real functions, the logarithm function was simply the inverse of the exponential function. Life is more complicated in the complex case—as we have seen, the complex exponential function is not invertible. There are many solutions to the equation $e^z = w$.

If $z \neq 0$, we define $\log z$ by

$$\log z = \ln|z| + i \arg z.$$

There are thus many $\log z$'s; one for each argument of z. The difference between any two of these is thus an integral multiple of $2\pi i$. First, for any value of $\log z$ we have

$$e^{\log z} = e^{\ln|z|+i\arg z} = e^{\ln|z|}e^{i\arg z} = z.$$

This is familiar. But next there is a slight complication:

$$\begin{aligned}\log(e^z) &= \ln e^x + i\arg e^z = x + (y+2k\pi)i \\ &= z + 2k\pi i,\end{aligned}$$

where k is an integer. We also have

$$\begin{aligned}\log(zw) &= \ln(|z||w|) + i\arg(zw) \\ &= \ln|z| + i\arg z + \ln|w| + i\arg w + 2k\pi i \\ &= \log z + \log w + 2k\pi i\end{aligned}$$

for some integer k.

There is defined a function, called the **principal logarithm**, or **principal branch** of the logarithm, function, given by

$$\operatorname{Log} z = \ln|z| + i\operatorname{Arg} z,$$

where $\operatorname{Arg} z$ is the principal argument of z. Observe that for any $\log z$, it is true that $\log z = \operatorname{Log} z + 2k\pi i$ for some integer k which depends on z. This new function is an extension of the real logarithm function:

$$\operatorname{Log} x = \ln x + i\operatorname{Arg} x = \ln x.$$

This function is analytic at a lot of places. First, note that it is not defined at $z = 0$, and is not continuous anywhere on the negative real axis ($z = x + 0i$, where $x < 0$.). So, let's suppose $z_0 = x_0 + iy_0$, where z_0 is not zero or on the negative real axis, and see about a derivative of $\operatorname{Log} z$:

$$\lim_{z\to z_0} \frac{\operatorname{Log} z - \operatorname{Log} z_0}{z - z_0} = \lim_{z\to z_0} \frac{\operatorname{Log} z - \operatorname{Log} z_0}{e^{\operatorname{Log} z} - e^{\operatorname{Log} z_0}}.$$

Now if we let $w = \operatorname{Log} z$ and $w_0 = \operatorname{Log} z_0$, and notice that $w \to w_0$ as $z \to z_0$, this becomes

$$\lim_{z\to z_0}\frac{\operatorname{Log} z - \operatorname{Log} z_0}{z - z_0} = \lim_{w\to w_0}\frac{w - w_0}{e^w - e^{w_0}}$$
$$= \frac{1}{e^{w_0}} = \frac{1}{z_0}$$

Thus, Log is differentiable at z_0 , and its derivative is $\frac{1}{z_0}$.

We are now ready to give meaning to z^c, where c is a complex number. We do the obvious and define

$$z^c = e^{c\log z}.$$

There are many values of $\log z$, and so there can be many values of z^c. As one might guess, $e^{c\operatorname{Log} z}$ is called the **principal value** of z^c.

Note that we are faced with two different definitions of z^c in case c is an integer. Let's see if we have anything to unlearn. Suppose c is simply an integer, $c = n$. Then

$$z^n = e^{n\log z} = e^{n(\operatorname{Log} z + 2k\pi i)}$$
$$= e^{n\operatorname{Log} z}e^{2kn\pi i} = e^{n\operatorname{Log} z}$$

There is thus just one value of z^n, and it is exactly what it should be: $e^{n\operatorname{Log} z} = |z|^n e^{in\arg z}$. It is easy to verify that in case c is a rational number, z^c is also exactly what it should be.

Far more serious is the fact that we are faced with conflicting definitions of z^c in case $z = e$. In the above discussion, we have assumed that e^z stands for $\exp(z)$. Now we have a definition for e^z that implies that e^z can have many values. For instance, if someone runs at you in the night and hands you a note with $e^{1/2}$ written on it, how to you know whether this means $\exp(1/2)$ or the two values $\sqrt{e}$ and $-\sqrt{e}$? Strictly speaking, you do not know. This ambiguity could be avoided, of course, by always using the notation $\exp(z)$ for $e^x e^{iy}$, but almost everybody in the world uses e^z with the understanding that this is $\exp(z)$, or equivalently, the principal value of e^z. This will be our practice.

Exercises

12. Is the collection of all values of $\log(i^{1/2})$ the same as the collection of all values of $\frac{1}{2}\log i$? Explain.

13. Is the collection of all values of $\log(i^2)$ the same as the collection of all values of $2\log i$? Explain.

14. Find all values of $\log(z^{1/2})$. (in rectangular form)

15. At what points is the function given by Log $(z^2 + 1)$ analytic? Explain.

16. Find the principal value of

a) i^i.

b) $(1 - i)^{4i}$

17. a)Find all values of $|i^i|$.