Integrals. Let $\vec{\boldsymbol{r}}(t) = (x(t), y(t), z(t)) : [a, b] \to \mathbb{R}^3$ parametrize a curve C.

Arclength. $\int_C ds = \int_a^b \|\vec{\boldsymbol{r}}'(t)\|\, dt$.

Line integral with respect to arclength. For a scalar-valued function $f : \mathbb{R}^3 \to \mathbb{R}$,

$$\int_C f\, ds = \int_a^b f(\vec{\boldsymbol{r}}(t)) \|\vec{\boldsymbol{r}}'(t)\|\, dt.$$

Line integral. For a vector field $\vec{\boldsymbol{F}} = (F_1, F_2, F_3) : \mathbb{R}^3 \to \mathbb{R}^3$,

$$\int_C \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}} = \int_C F_1\, dx + F_2\, dy + F_3\, dz = \int_a^b \vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}(t)) \cdot \vec{\boldsymbol{r}}'(t)\, dt.$$

Integral Theorems in the Plane. Let $\vec{\boldsymbol{F}} = (F_1, F_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field, that is smooth on a bounded region R in $\mathbb{R}^2$. Let C denote the boundary curve of R, oriented in the standard way.

Green's Theorem.

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy = \oint_C F_1\, dx + F_2\, dy = \oint_C \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}.$$

Area Theorem. In particular,

$$Area(R) = \iint_R dx\, dy = \oint_C x\, dy = -\oint_C y\, dx = \frac{1}{2} \oint_C x\, dy - y\, dx.$$

More Integrals. Let R be a subset of $\mathbb{R}^2$, and let $\vec{\boldsymbol{r}}(u, v) = (x(u, v), y(u, v), z(u, v)) : R \to S$ parametrize a surface $S \subseteq \mathbb{R}^3$.

Surface area. $\iint_S dA = \iint_R \|\vec{\boldsymbol{r}}_{\boldsymbol{u}} \times \vec{\boldsymbol{r}}_{\boldsymbol{v}}\| du\, dv$.

Surface integral. For a scalar-valued function $f : S \to \mathbb{R}$,

$$\iint_S f\, dA = \iint_R f(\vec{\boldsymbol{r}}(u, v)) \|\vec{\boldsymbol{r}}_{\boldsymbol{u}} \times \vec{\boldsymbol{r}}_{\boldsymbol{v}}\| du\, dv$$

Flux integral. Let $\vec{\boldsymbol{F}} = (F_1, F_2, F_3) : S \to \mathbb{R}^3$ be a vector field, and let $\vec{\boldsymbol{n}}$ be a choice of unit normal for the orientable surface S. Then

$$\iint_S \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{n}}\, dA \quad = \quad \iint_R \vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}(u, v)) \cdot (\vec{\boldsymbol{r}}_{\boldsymbol{u}} \times \vec{\boldsymbol{r}}_{\boldsymbol{v}})\, du\, dv$$

Integral Theorems in $\mathbb{R}^3$.

Gauss' Divergence Theorem. Let T be a closed bounded solid in $\mathbb{R}^3$ with boundary surface ∂T, oriented such that its unit normal points outward. Let $\vec{\boldsymbol{F}} = (F_1, F_2, F_3) : T \to \mathbb{R}^3$ be a vector field.

$$\iiint_T div\, \vec{\boldsymbol{F}}\, dV = \iiint_T (\nabla \cdot \vec{\boldsymbol{F}})\, dV = \iint_{\partial T} \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{n}}\ dA.$$

Stokes' Theorem. Let $S \subseteq \mathbb{R}^3$ be an orientable surface with the simple closed curve ∂S as its boundary. Let $\vec{\boldsymbol{F}} = S \to \mathbb{R}^3$ be a vector field. Then

$$\iint_S (curl\, \vec{\boldsymbol{F}}) \cdot \vec{\boldsymbol{n}}\, dA = \iint_S (\nabla \times \vec{\boldsymbol{F}}) \cdot \vec{\boldsymbol{n}}\, dA = \oint_{\partial S} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}.$$

Here $\vec{\boldsymbol{n}}$ is a choice of unit normal for S, and ∂S is being traversed such that $\vec{\boldsymbol{n}}$ is to the left of the curve ∂S.