Math 4303

Worksheet #1

January 18, 2018

Suppose that addition and multiplication have already been defined for the set of natural numbers \mathbb{N} .

1. We define a relation \sim on $\mathbb{N} \times \mathbb{N}$ as follows:

$$(p,q) \sim (p',q') \Leftrightarrow p+q'=p'+q.$$

Show that \sim defines an equivalence relation on $\mathbb{N} \times \mathbb{N}$.

2. It then makes sense to define equivalence classes $(p,q)_{\sim}$:

$$(p,q)_{\sim} := \{ (p',q') \in \mathbb{N} \times \mathbb{N} \mid (p',q') \sim (p,q) \}.$$

The set of all these equivalence classes is denoted by $(\mathbb{N} \times \mathbb{N})_{\sim}$.

Find all elements in the equivalence class $(2,5)_{\sim}$. What do all these pairs of natural numbers have in common?

Find all elements in the equivalence class $(4, 2)_{\sim}$. What do all these pairs of natural numbers have in common?

- 3. One can then identify the set of integers Z with this set (N×N)_∼. Which equivalence class corresponds to the integer 0? What about the equivalence classes corresponding to the integers 1 and -3, respectively?
- 4. How can one define addition of two integers? More precisely, what should be the meaning of

$$(p,q)_{\sim} + (p',q')_{\sim}?$$

Is your definition well-defined¹?

- 5. Show that addition as defined in 4. is commutative.
- 6. What is the neutral element in $(\mathbb{N} \times \mathbb{N})_{\sim}$ with respect to addition?
- 7. Given $(p,q)_{\sim} \in (\mathbb{N} \times \mathbb{N})_{\sim}$, what is the inverse element of $(p,q)_{\sim}$ with respect to addition?
- 8. How can one define multiplication of two integers? More precisely, what should be the meaning of

$$(p,q)_{\sim} \cdot (p',q')_{\sim}?$$

Is your definition well-defined?

9. Verify that $(2,5)_{\sim} \cdot (1,2)_{\sim} = (5,2)_{\sim}$.

 $^{^{1}}$ You have to check that your definition does not depend on the representatives chosen from the two equivalence classes.