3 Sequences and Series of Functions

3.1 Pointwise and Uniform Convergence

We now turn our attention to the convergence of sequences and series of functions. Here is the natural definition to extend the notion of convergence from numbers to functions:

Let $D \subseteq \mathbb{R}$. Given functions $f_n : D \to \mathbb{R}$ and $f : D \to \mathbb{R}$, we say the sequence $(f_n)_{n\in\mathbb{N}}$ *converges to f pointwise* if $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for all $x \in D$.

Equivalently, this means that for all $x \in D$ and for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$ it follows that $|f_n(x) - f(x)| < \varepsilon$.

Exercise 3.1
Let $f_n : [0,1] \to \mathbb{R}$ be given by $f(x) = x^n$. Find a suitable $f : [0,1] \to \mathbb{R}$ such that (f_n) converges to f pointwise.

This example reveals the first deficiency of pointwise convergence: the pointwise limit of a sequence of continuous functions is not necessarily continuous.

The next example shows that the pointwise limit of a sequence of bounded functions is not necessarily bounded. We say a function $f : D \to \mathbb{R}$ is *bounded* if there is an $M > 0$ such that $|f(x)| \leq M$ for all $x \in D$.

Exercise 3.2
Let $f_n : (-1,1) \to \mathbb{R}$ be given by $f_n(x) = \sum\limits_{k=0}^{n} x^k$. Show that (f_n) converges pointwise to the function $f(x) = \dfrac{1}{1-x}$.

Let us say that a sequence $f_n : D \to \mathbb{R}$ is *uniformly bounded*, if there is an $M > 0$ such that $|f_n(x)| \leq M$ for all $x \in D$ and $n \in \mathbb{N}$.

Assuming this extra assumption, we obtain a positive result:

Exercise 3.3
Suppose the sequence $f_n : D \to \mathbb{R}$ is uniformly bounded and converges pointwise to the function f. Then f is bounded.

Pointwise convergence also does not interact nicely with Riemann integration:

Exercise 3.4
Let $f_n : [0,1] \to \mathbb{R}$ be given by

$$f_n(x) = \begin{cases} 0, \text{ if } x = 0 \\ n, \text{ if } 0 < x \leq \frac{1}{n} \\ 0, \text{if } \frac{1}{n} < x \leq 1 \end{cases}$$

Show that this sequence converges pointwise to the zero-function.

Observe that $\displaystyle\int_0^1 f_n(x)\,dx = 1$ for all n, while the pointwise limit has integral 0.

We have already seen in Exercise 3.1 that the pointwise limit of differentiable functions is not necessarily differentiable. The next example shows that even in the case when the pointwise limit is differentiable, its derivative does not necessarily have the desired properties.

Task 3.5
Let $f_n : [-1,1] \to \mathbb{R}$ be defined by $f_n(x) = \dfrac{x}{1+nx^2}$. Show that each f_n is differentiable, that (f_n) has as its pointwise limit f the zero function, but that $\lim\limits_{n\to\infty} f_n'(0) \neq f'(0)$.

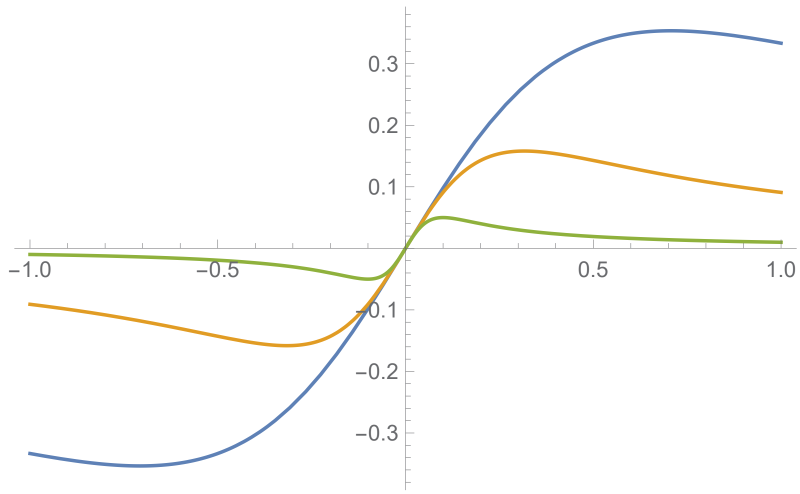

Figure 1: The functions f_2, f_{10} and f_{100} from Task 3.5

All these examples show that pointwise convergence is not such a useful property[14].

We will therefore study a different limit concept for functions:

Let $D \subseteq \mathbb{R}$. Given functions $f_n : D \to \mathbb{R}$ and $f : D \to \mathbb{R}$, we say the sequence $(f_n)_{n\in\mathbb{N}}$ *converges to f uniformly* if for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$ and for all $x \in D$ it follows that $|f_n(x) - f(x)| < \varepsilon$.

Exercise 3.6
Let functions $f_n : D \to \mathbb{R}$ and $f : D \to \mathbb{R}$ be given. If (f_n) converges to f uniformly, then (f_n) converges to f pointwise.

Exercise 3.7
Show that the converse of Exercise 3.6 is false.

[14] Actually, in the case of integration, this led to the development of a different notion of integration: the Lebesgue integral with its Dominated Convergence Theorem.

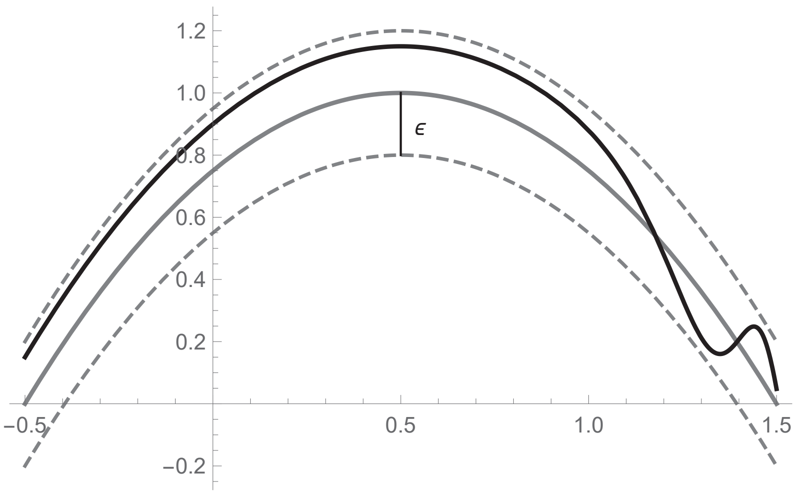

Figure 2: Uniform convergence: The function f_n (in black) lies in an ε-tube around the function f (in gray).

Exercise 3.8
Let a sequence of functions $f_n : D \to \mathbb{R}$ be given. The sequence (f_n) converges uniformly if and only if for all $\varepsilon > 0$ there is an $n \in \mathbb{N}$ such that for all $x \in D$ and for all $m, n \geq N$ it follows that $|f_m(x) - f_n(x)| < \varepsilon$.

Exercise 3.9
Let a sequence of functions $f_n : D \to \mathbb{R}$ and numbers $M_n \geq 0$ be given. Suppose

$$|f_n(x)| \leq M_n \text{ for all } x \in D \text{ and } n \in \mathbb{N}.$$

Show: If $\sum M_n$ converges, then $\sum f_n$ converges uniformly (and absolutely).

The next two results highlight some of the permanence properties of uniform convergence:

Task 3.10
Let $f_n : D \to \mathbb{R}$ be continuous functions such that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to some function f. Then f is continuous.

Task 3.11
Let $f_n : D \to \mathbb{R}$ be bounded functions such that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to some function f. Then f is bounded.

There are two more permanence results for uniform convergence:

Theorem 3.1. Let $f_n : [a,b] \to \mathbb{R}$ be Riemann-integrable functions such that (f_n) converges uniformly to some function f. For $t \in [a,b]$, let $F_n(t) = \displaystyle\int_a^t f_n(x)\,dx$.

Then f is Riemann integrable, and moreover (F_n) converges uniformly to the function F, defined by $F(t) = \displaystyle\int_a^t f(x)\,dx$.

Theorem 3.2. Let $f_n : [a,b] \to \mathbb{R}$ be differentiable functions such that (f_n') converges uniformly to some function g. Assume additionally that for some $x_0 \in [a,b]$ the sequence $(f_n(x_0))$ converges.

Then (f_n) converges uniformly to some function f, f is differentiable on $[a,b]$, and $f'(x) = g(x)$ for all $x \in [a,b]$.