Theorem. If there is an infinite set, then there is a model for the natural numbers.

Let A be an infinite set. Then there is a function $S : A \to A$ that is injective, but not surjective. Thus we can find an $a_0 \in A$ with $a_0 \notin S(A)$. Let

$$\mathcal{K} = \{B \subseteq A \mid a_0 \in B \text{ and } S(B) \subseteq B\}$$

Note that $A \in \mathcal{K}$, so $\mathcal{K} \neq \emptyset$. We set

$$N = \bigcap_{B \in \mathcal{K}} B.$$

Observe that $N \in \mathcal{K}$. Indeed, $a_0 \in N$, since $a_0 \in B$ for all $B \in \mathcal{K}$. Also

$$S(N) = S\left(\bigcap_{B \in \mathcal{K}} B\right) \subseteq \bigcap_{B \in \mathcal{K}} S(B) \subseteq \bigcap_{B \in \mathcal{K}} B = N.$$

By its definition the set N is thus the smallest element of $\mathcal{K}$.

Finally we show that N with the function $S : N \to N$ (as successor function) and a_0 (in the role of 0) satisfies Axioms **D1–D3**.

As the restriction of the injective function $S : A \to A$ to N, the function $S : N \to N$ is also injective. Thus **D1** is satisfied.

For **D2** we have to show that $S(N) = N \setminus \{a_0\}$. Since $a_0 \notin S(N)$ and $S(N) \subseteq N$, we obtain that $S(N) \subseteq N \setminus \{a_0\}$. For the remaining subset relation suppose to the contrary that there is a second element missing from the range of N: there is an element $n_0 \in N$ satisfying $n_0 \notin S(N)$ and $n_0 \neq a_0$. Set $N_0 = N \setminus \{n_0\}$. Note that $a_0 \in N_0$ and that $S(N_0) \subseteq N_0$. Thus $N_0 \in \mathcal{K}$. We also know that $N_0 \subsetneqq N$, yielding a contradiction.

Now let $M \subseteq N$, with $a_0 \in M$, and satisfying $S(M) \subseteq M$. Then $M \in \mathcal{K}$, and thus, again using the minimality of N in $\mathcal{K}$, it follows that $M \supseteq N$. This proves **D3**.