compact set has a minimum. Take a norm
ct set be the unit sphere for the sup norm.

VII, §2.) Let $E = \mathbf{R}^k$ and let S be a closed
there exists a point $w \in S$ such that

$$) = |w - v|.$$

suitable radius, centered at v, and consider
.]

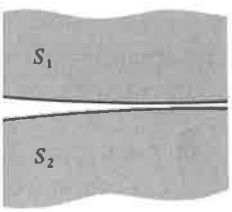
be a closed subset of $\mathbf{R}^k$. Define

$$= \operatorname{glb}_{\substack{x \in K \\ y \in S}} |x - y|.$$

nd $y_0 \in S$ such that

$$= |x_0 - y_0|.$$

$\mapsto d(S, x)$ for $x \in K$.]
mpact, then the conclusion does not neces-
two sets S_1 and S_2:



se distance is the distance between the sets,
proach each other.)

$\to K$ be a continuous map. Suppose that

$$(y)| \geqq |x - y|$$

e inverse map $f^{-1}: f(K) \to K$ is continuous.
en $x_0 \in K$, consider the sequence $\{f^n(x_0)\}$,
u might use Corollary 2.3.]

w that there exists a sequence of compact
$_{j+1})$ for all j, and such that the union of all
l ball of radius j, and let K_j be the set of
$\geqq 1/j$.]

VIII, §3. ALGEBRAIC CLOSURE OF THE COMPLEX NUMBERS

A polynomial with complex coefficients is simply a complex valued function f of complex numbers which can be written in the form

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n, \qquad a_i \in \mathbf{C}.$$

We call $a_0, \ldots, a_n$ the coefficients of f, and these coefficients are uniquely determined, just as in the real case. If $a_n \neq 0$, we call n the **degree** of f. A **root** of f is a complex number z_0 such that $f(z_0) = 0$. *To say that the complex numbers are* **algebraically closed** *is, by definition, to say that every polynomial of degree* $\geqq 1$ *has a root in* $\mathbf{C}$. *We shall now prove that this is the case.*

We write

$$f(t) = a_n t^n + \cdots + a_0$$

with $a_n \neq 0$. For every real number R, the function $|f|$ such that

$$t \mapsto |f(t)|$$

is continuous on the closed disc of radius R, which is compact. Hence this function (real valued!) has a minimum value on this disc. On the other hand, from the expression

$$f(t) = a_n t^n \left(1 + \frac{a_{n-1}}{a_n t} + \cdots + \frac{a_0}{a_n t^n}\right)$$

we see that when $|t|$ becomes large, $|f(t)|$ also becomes large, i.e. given $C > 0$, there exists $R > 0$ such that if $|t| > R$ then $|f(t)| > C$. Consequently, there exists a positive number R_0 such that, if z_0 is a minimum point of $|f|$ on the closed disc of radius R_0, then

$$|f(t)| \geqq |f(z_0)|$$

for all complex numbers t. In other words, z_0 is an absolute minimum of $|f|$. We shall prove that $f(z_0) = 0$.

We express f in the form

$$f(t) = c_0 + c_1(t - z_0) + \cdots + c_n(t - z_0)^n$$

with constants c_i. If $f(z_0) \neq 0$, then $c_0 = f(z_0) \neq 0$. Let $z = t - z_0$ and let m be the smallest integer >0 such that $c_m \neq 0$. This integer m

exists because f is assumed to have degree $\geqq 1$. Then we can write

$$f(t) = f_1(z) = c_0 + c_m z^m + z^{m+1} g(z)$$

for some polynomial g, and some polynomial f_1 (obtained from f by changing the variable). Let z_1 be a complex number such that

$$z_1^m = -c_0/c_m,$$

and consider values of z of the type $z = \lambda z_1$, where λ is real, $0 \leqq \lambda \leqq 1$. We have

$$\begin{aligned} f(t) = f_1(\lambda z_1) &= c_0 - \lambda^m c_0 + \lambda^{m+1} z_1^{m+1} g(\lambda z_1) \\ &= c_0[1 - \lambda^m + \lambda^{m+1} z_1^{m+1} c_0^{-1} g(\lambda z_1)]. \end{aligned}$$

There exists a number $C > 0$ such that for all λ with $0 \leqq \lambda \leqq 1$ we have

$$|z_1^{m+1} c_0^{-1} g(\lambda z_1)| \leqq C$$

(continuous function on a compact set), and hence

$$|f_1(\lambda z_1)| \leqq |c_0|(1 - \lambda^m + C\lambda^{m+1}).$$

If we can now prove that for sufficiently small λ with $0 < \lambda < 1$ we have

$$0 < 1 - \lambda^m + C\lambda^{m+1} < 1,$$

then for such λ we get $|f_1(\lambda z_1)| < |c_0|$, thereby contradicting the hypothesis that $|f(z_0)| \leqq |f(t)|$ for all complex numbers t. The left-hand inequality is of course obvious since $0 < \lambda < 1$. The right-hand inequality amounts to $C\lambda^{m+1} < \lambda^m$, or equivalently $C\lambda < 1$, which is certainly satisfied for sufficiently small λ. This concludes the proof.

Remark. The idea of the proof is quite simple. We have our polynomial

$$f_1(z) = c_0 + c_m z^m + z^{m+1} g(z),$$

and $c_m \neq 0$. If $g = 0$, we simply adjust $c_m z^m$ so as to subtract a term in the same direction as c_0, to shrink c_0 toward the origin. This is done by extracting the suitable m-th root as above. Since $g \neq 0$ in general, we have to do a slight amount of juggling to show that the third term is very small compared to $c_m z^m$, and that it does not disturb the general idea of the proof in an essential way.