The problems are due on September 11.

For all students:

Problem 1 Let A be a subset of a metric space (X, d). Show that the following two definitions are equivalent:

- The interior of A is the set of all interior points of A.
- The interior of A is the union of all open sets contained in A.

Problem 2 Let Y be a subset of the metric space (X, d).

a. Show that (Y, d) is also a metric space.

b. Show that a set A is open in Y if and only if there is an open set O in X such that $A = O \cap Y$.

Problem 3 Let C be the space of all real-valued continuous functions on $[0, 1]$, endowed with the norm $\|f\| = \max_{x\in[0,1]} |f(x)|$. Recall that $(C, \|\cdot\|)$ is a normed vector space. Let

$$A = \{f \in C \mid f(1/3) \neq 0\}$$

a. Is A open? Is it closed?

b. Find the interior, the closure and the boundary of A.

Problem 4 Let X be a $\mathbb{R}$-vector space, and let $\langle\cdot,\cdot\rangle$ be an inner product on X. As usual, we set $\|x\| = \sqrt{\langle x, x\rangle}$ for $x \in X$. Show the following hold for all $x, y \in X$:

a. $2\|x\|^2 + 2\|y\|^2 = \|x + y\|^2 + \|x - y\|^2$

b. $\|x + y\|\|x - y\| \leq \|x\|^2 + \|y\|^2$

c. $4\langle x, y\rangle = \|x + y\|^2 - \|x - y\|^2$

For graduate students:

Problem 5 Let $(X, \|\cdot\|)$ be a normed $\mathbb{R}$-vector space such that

$$2\|x\|^2 + 2\|y\|^2 = \|x + y\|^2 + \|x - y\|^2 \tag{1}$$

holds for all $x, y \in X$. Identity (1) is called the "parallelogram identity". Show that

$$\langle x, y\rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2)$$

then defines an inner product on X.