Homework 4 **Analysis II** **Fall 2018**

The problems are due on Tuesday, October 16.

For all students:

Problem 1 Let $(X, \|\cdot\|)$ be a normed vector space. Show that the only subsets of X that are both open and closed are X and the empty set.

Problem 2 Let A be a countable subset of $\mathbb{R}^2$. Show that $\mathbb{R}^2 \setminus A$ is path-connected.

Problem 3

1. Suppose A is a connected subset of the metric space (X, d). Show that B is connected if $A \subseteq B \subseteq cl(A)$.
2. Show: If B_α is a connected subset of a metric space (X, d) for all $\alpha \in A$, and $B_\alpha \cap B_\beta \neq \emptyset$ for all $\alpha, \beta \in A$, then $\bigcup_{\alpha \in A} B_\alpha$ is connected.

Problem 4

1. Let (X, d) be a countable non-empty metric space. Let $x \in X$. Show that there is a real number $r > 0$, such that
$$\{z \in X \mid d(x, z) = r\} = \emptyset.$$
2. Let (X, d) be a connected metric space with at least two elements. Show that X is uncountable.

For graduate students:

Problem 5 Let (X, d) be a metric space.

1. We say two Cauchy sequences (x_n) and (y_n) in X are *equivalent* if $\lim_{n\to\infty} d(x_n, y_n) = 0$. Show that this is indeed an equivalence relation.
2. Let $X^\sharp$ be the set of all equivalence classes of Cauchy sequences obtained in this way. If $R, S \in X^\sharp$ with $(r_n) \in R$, and $(s_n) \in S$, define $D(R, S) = \lim_{n\to\infty} d(r_n, s_n)$. Note that the limit exists by a previous homework problem. Show that D is well-defined (i.e. the definition of $D(R, S)$ is independent of the chosen representatives (r_n) and (s_n) in the two equivalence classes R and S).
3. Show that $(X^\sharp, D)$ is a metric space.
4. Show that $(X^\sharp, D)$ is complete.