Homework 5 **Analysis II** **Fall 2018**

The problems are due on Thursday, October 25.

For all students:

Problem 1 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as follows:

$$f(x,y) = \begin{cases} 0, & \text{if } (x,y) = (0,0) \\ \dfrac{xy^2}{x^2+y^4}, & \text{if } (x,y) \neq (0,0) \end{cases}$$

1. Show that f is bounded. (This means there is an $M \in \mathbb{R}$ such that $|f(x,y)| \leq M$ for all $(x,y) \in \mathbb{R}^2$.)
2. Let $(x_0, y_0) \neq (0,0)$. Define $\ell : \mathbb{R} \to \mathbb{R}^2$ by $\ell(t) = (tx_0, ty_0)$. Show that $f \circ \ell : \mathbb{R} \to \mathbb{R}$ is continuous at 0. (This means that the function f is continuous at $(0,0)$ "in every direction.")
3. Show that f is not continuous at $(0,0)$.

Problem 2 Let X and Y be two metric spaces. Show that $f : X \to Y$ is continuous if and only if for all subsets A, B of X with $cl(A) = cl(B)$ in X, their images $f(A)$ and $f(B)$ satisfy $cl(f(A)) = cl(f(B))$ in Y.

Problem 3 Let (X,d) be a compact metric space, and $f : X \to X$ be a map satisfying

$$d(f(x), f(y)) = d(x,y) \text{ for all } x, y \in X.$$

Show that f is a bijection. (Hint: Denote by $f^n := \underbrace{f \circ f \ldots \circ f}_{n \text{ times}}$ the n-fold composition of f with itself. If $y \notin f(X)$, consider the sequence $(f^n(y))$.)

Problem 4 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$. Assume also that $\lim_{|x|\to\infty} f(x) = 0$. (This means that for all $\varepsilon > 0$ there is an $N \in \mathbb{R}$ such that $|f(x)| < \varepsilon$ whenever $|x| > N$.) Show that f is uniformly continuous on $\mathbb{R}$.

Problem 5 Let (X,d) and (Y,ρ) be metric spaces and $D \subset X$.

1. Suppose $f : D \to Y$ is uniformly continuous on D. Show: If (x_n) is a Cauchy sequence in D, then $(f(x_n))$ is a Cauchy sequence in Y.
2. Prove or disprove the converse to 1.
3. Show that the result in 1. fails if one only assumes that f is continuous on D.