Homework 7 **Analysis II** **Fall 2018**

The problems are due on Tuesday, December 4.

For all students:

Problem 1 A function $f : \mathbb{R} \to \mathbb{R}$ is called *lower semi-continuous* if whenever $x \in \mathbb{R}$ and $\alpha \in \mathbb{R}$ satisfy $f(x) > \alpha$, there exists a $\delta > 0$ such that $f(y) > \alpha$ for all $y \in \mathbb{R}$ satisfying $|x - y| < \delta$.

1. Observe that $f : \mathbb{R} \to \mathbb{R}$, given by

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise,} \end{cases}$$

   is lower semi-continuous, but not continuous.

2. Show: If a sequence of lower semi-continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ satisfies

   (a) $f_{n+1}(x) \geq f_n(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, and

   (b) (f_n) converges pointwise to a function f,

   then f is also lower semi-continuous.

Problem 2 Let $\mathcal{B}$ be a compact subset of $C(0,1)$, the space of continuous functions $f : [0,1] \to \mathbb{R}$, with the norm $\|f\| = \sup\{|f(x)| : x \in [0,1]\}$. Show that $\mathcal{B}$ is equicontinuous and pointwise compact.

Problem 3 Let X be a compact metric space, and $f, f_n : X \to \mathbb{R}$ be continuous functions such that

1. $f_{n+1}(x) \geq f_n(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, and

2. (f_n) converges pointwise to f.

Then (f_n) converges to f uniformly.

Problem 4 The problem will establish the existence of a continuous function that is not differentiable anywhere. Let $f_0 : \mathbb{R} \to \mathbb{R}$ be the 1-periodic function defined as follows:

$$f_0(x) = \begin{cases} x & , \text{if } 0 \leq x - \lfloor x \rfloor \leq \frac{1}{2} \\ 1 - x & , \text{if } \frac{1}{2} < x - \lfloor x \rfloor < 1 \end{cases}$$

Here $\lfloor x \rfloor$ denotes the greatest integer function. For $n \in \mathbb{N}$, we define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = 2^{-n} f_0(2^n x).$$

Show the following:

1. $\sum f_k$ converges uniformly on $[0,1]$. The function $g = \sum_{k=1}^{\infty} f_k$ is continuous on $[0,1]$.

2. Let $f : [0,1] \to \mathbb{R}$ be a function that is differentiable at the point $y \in (0,1)$. Then for all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\left| \frac{f(z) - f(x)}{z - x} - f'(y) \right| < \varepsilon$$

for all $x, z \in [0,1]$ satisfying $x \leq y \leq z$, $x \neq y$, $|y - x| < \delta$ and $|z - y| < \delta$.

3. For all $y \in (0,1)$ there are four sequences (x_n), (x'_n), (z_n), and (z'_n) in $[0,1]$ with the following properties:

   (a) All four sequences converge to y.

   (b) $x_n \leq y \leq z_n$, $x_n \neq z_n$ for all $n \in \mathbb{N}$.

   (c) $x'_n \leq y \leq z'_n$, $x'_n \neq z'_n$ for all $n \in \mathbb{N}$.

   (d) $\left| \frac{g(z_n) - g(x_n)}{z_n - x_n} - \frac{g(z'_n) - g(x'_n)}{z'_n - x'_n} \right| \geq 1$ for all $n \in \mathbb{N}$.

   Hint: Draw the graphs of $\sum_{k=1}^{n} f_k$ *for small* n.

4. *Use the results above to show that the function* g *fails to be differentiable at all points in* $(0,1)$.

For graduate students:

Problem 5 Let $(f_n) : \mathbb{R} \to [0,1]$ be a sequence of *increasing* functions, i.e. satisfying $f_n(x) \leq f_n(y)$ for all $x \leq y$ and all $n \in \mathbb{N}$. The problem will establish that a subsequence of (f_n) converges pointwise on $\mathbb{R}$.

1. Show that there is a subsequence (f_{n_k}) of (f_n), which converges at all rational points in $\mathbb{R}$, say $\lim_{k\to\infty} f_{n_k}(q) =: f(q)$ for all rational numbers q.

2. For $x \in \mathbb{R}$ define $f(x) := \sup\{f(q) \mid q \leq x \text{ , } q \text{ is rational}\}$. Using the monotonicity of the functions involved, show that $f(x) = \lim_{k\to\infty} f_{n_k}(x)$ for all $x \in \mathbb{R}$, at which f is continuous.

3. Show that a further subsequence of (f_{n_k}) converges to f for *all* $x \in \mathbb{R}$.
   Hint: How many points are there, at which f *is discontinuous?*