Reflective thinking turns experience into insight.
John Maxwell

1 Numbers

In 1879, Gottlob Frege completed the first step of his program to put mathematics on a solid foundation. His idea was that **logic** should be the foundation of all mathematics, and, following Gottfried von Leibniz (1646–1716) and George Boole (1815–1864), he created a rigorous symbolic language, which he called *Begriffs-schrift*, to incorporate all standard principles of logic.

Georg Cantor followed in his footsteps and developed **set theory** from basic logical principles. In 1888, Richard Dedekind took the next step, and presented a **construction of the real numbers** based on set theory.

It should be mentioned that Frege's program was doomed to fail. Frege's construction allowed objects such as "the set of all sets". Bertrand Russell used this to construct a paradox: Let E denote the set of all sets which do not contain themselves as members. Is E an element of E? It can't be, because E contains only sets which are **not** members of themselves. Can E fail to be an element of E? No, since if $E \notin E$, then by the definition of the set E, E is contained in E.

Bertrand Russell's and Alfred Whitehead's attempts to "fix" these problems in their monumental *Principia Mathematica* are generally regarded as artificial and therefore in violation of the spirit of Frege's program.

In response, David Hilbert came up with an alternative program: Use axiomatic systems as the foundation of mathematics together with *meta-mathematics*. Mathematicians "do" mathematics starting from axiomatic systems; meta-mathematics allows to talk about the process "from the outside" addressing issues such as completeness[1] and consistency[2] of a given axiomatic system.

In 1930, Kurt Gödel showed that this approach was equally flawed: It is not possible to show (within the axiomatic system) that an axiomatic system which incorporates the arithmetic of natural numbers is complete (or consistent).

[1] An axiomatic system is complete, if all statements within the axiomatic system can—in principle—be shown to be true or to be false.

[2] An axiomatic system is said to be consistent, if the axioms can be shown not to lead to contradictions.

1.1 The Natural Numbers

Definition. Richard Dedekind started by giving the following definition of the set of **Natural Numbers**[3]:

> *The natural numbers are a set $\mathbb{N}$ together with a special element called* 0*, and a function $S : \mathbb{N} \to \mathbb{N}$ satisfying the following axioms:*
>
> **(D1)** *S is injective*[4].
>
> **(D2)** $S(\mathbb{N}) = \mathbb{N} \setminus \{0\}$.[5]
>
> **(D3)** *If a subset M of $\mathbb{N}$ contains 0 and satisfies $S(M) \subseteq M$, then $M = \mathbb{N}$.*

The function S is called the successor function.

The first two axioms describe the process of counting, the third axiom assures the **Principle of Induction**:

Exercise 1.1

Let $P(n)$ be a predicate with the set of natural numbers as its domain. If

1. $P(0)$ is true, and
2. $P(S(n))$ is true, whenever $P(n)$ is true,

then $P(n)$ holds for all natural numbers.

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[3] A similar definition of the natural numbers was introduced by Giuseppe Peano in 1889:

> *The natural numbers are a set $\mathbb{N}$ together with a special element called* 0*, and a function $S : \mathbb{N} \to \mathbb{N}$ satisfying the following axioms:*
>
> **(P1)** $0 \in \mathbb{N}$.
>
> **(P2)** *If $n \in \mathbb{N}$, then $S(n) \in \mathbb{N}$.*
>
> **(P3)** *If $n \in \mathbb{N}$, then $S(n) \neq 0$.*
>
> **(P4)** *If a set A contains* 0*, and if A contains $S(n)$, whenever it contains n, then the set A contains $\mathbb{N}$.*
>
> **(P5)** *$S(m) = S(n)$ implies $m = n$ for all $m, n \in \mathbb{N}$.*

[4] A function $f : A \to B$ is called *injective* if for all $a_1, a_2 \in A$, $f(a_1) = f(a_2)$ implies $a_1 = a_2$.

[5] For a function $f : A \to B$, $f(A) := \{b \in B \mid f(a) = b \text{ for some } a \in A\}$.

Arithmetic Properties. **Addition** of natural numbers is established recursively in the following way: For a fixed but arbitrary $m \in \mathbb{N}$ we define

$$\begin{aligned} m + 0 &:= m \\ m + S(n) &:= S(m+n) \qquad \text{for all } n \in \mathbb{N} \end{aligned}$$

Exercise 1.2
If we set $S(0) := 1$, then $S(m) = m + 1$ for all natural numbers $m \in \mathbb{N}$.

Use induction for the following:

Exercise 1.3
Show that addition on $\mathbb{N}$ is associative.

Exercise 1.4
Show that addition on $\mathbb{N}$ is commutative.

This last exercise implies in particular that 0 is the (unique) neutral element with respect to addition: $n + 0 = 0 + n = n$ holds for all $n \in \mathbb{N}$.

Multiplication of natural numbers is also defined recursively as follows: For a fixed but arbitrary $m \in \mathbb{N}$ we define

$$\begin{aligned} m \cdot 0 &:= 0 \\ m \cdot (n+1) &:= m \cdot n + m \qquad \text{for all } n \in \mathbb{N} \end{aligned}$$

Exercise 1.5
Show that the following distributive law holds for natural numbers:

$$(m + n) \cdot k = m \cdot k + n \cdot k.$$

Exercise 1.6
Show that 1 is the neutral element with respect to multiplication: For all natural numbers m,

$$m \cdot 1 = 1 \cdot m = m.$$

Exercise 1.7
Show that multiplication on $\mathbb{N}$ is commutative.

Exercise 1.8
Show that multiplication on $\mathbb{N}$ is associative.

Exercise 1.9
Show that multiplication is zero-divisor free:

$$m \cdot n = 0 \text{ implies } m = 0 \text{ or } n = 0.$$

Finally we can impose a **total order**[6] on $\mathbb{N}$ as follows: We say that $m \leq n$, if there is a natural number k, such that $m + k = n$.

Show that "$\leq$" is indeed a total order:

Exercise 1.10
"$\leq$" is reflexive[7].

Task 1.11
"$\leq$" is anti-symmetric[8].

Exercise 1.12
"$\leq$" is transitive[9].

Task 1.13
For all $m, n \in \mathbb{N}$, $m \leq n$ or $n \leq m$.

[6] A relation $\sim$ on A is called a *total order*, if $\sim$ is reflexive, anti-symmetric, transitive, and has the property that for all $a, b \in A$, $a \sim b$ or $b \sim a$ holds.

[7] A relation $\sim$ on A is *reflexive* if for all $a \in A$, $a \sim a$.

[8] A relation $\sim$ on A is *anti-symmetric* if for all $a, b \in A$ the following holds: $a \sim b$ and $b \sim a$ implies that a=b.

[9] A relation $\sim$ on A is *transitive* if for all $a, b, c \in A$ the following holds: $a \sim b$ and $b \sim c$ implies that $a \sim c$.

Show the following two compatibility laws:

Task 1.14
If $m \leq n$, then $m + k \leq n + k$ for all $k \in \mathbb{N}$.

Task 1.15
If $m \leq n$, then $m \cdot k \leq n \cdot k$ for all $k \in \mathbb{N}$.

Infinite Sets and the Existence of the Set of Natural Numbers. Do natural numbers exist? Following Dedekind, we will say that a set M is **infinite**, if there is an injective map $f : M \to M$ that is not surjective[10].

Exercise 1.16
Show that the set of natural numbers as defined on p. 2 is infinite.

Thus, the existence of the set of natural numbers implies the existence of infinite sets. In fact, we will show that the converse also holds:

Theorem. If there is an infinite set, then there is a model for the natural numbers.

Proof: Let A be an infinite set. Then there is a function $S : A \to A$ that is injective, but not surjective. Thus we can find an $a_0 \in A$ with $a_0 \notin S(A)$. Let

$$\mathcal{K} = \{B \subseteq A \mid a_0 \in B \text{ and } S(B) \subseteq B\}$$

[10] A function $f : A \to B$ is called *surjective*, if $f(A) = B$.

Note that $A \in \mathcal{K}$, so $\mathcal{K} \neq \emptyset$. We set

$$N = \bigcap_{B \in \mathcal{K}} B.$$

Observe that $N \in \mathcal{K}$. Indeed, $a_0 \in N$, since $a_0 \in B$ for all $B \in \mathcal{K}$. Also

$$S(N) = S\left(\bigcap_{B \in \mathcal{K}} B\right) \subseteq \bigcap_{B \in \mathcal{K}} S(B) \subseteq \bigcap_{B \in \mathcal{K}} B = N.$$

By its definition the set N is thus the smallest element of $\mathcal{K}$.

Finally we show that N with the function $S : N \to N$ (as successor function) and a_0 (in the role of 0) satisfies Axioms (D1)–(D3).

As the restriction of the injective function $S : A \to A$ to N, the function $S : N \to N$ is also injective. Thus (D1) is satisfied.

For (D2) we have to show that $S(N) = N \setminus \{a_0\}$. Since $a_0 \notin S(N)$ and $S(N) \subseteq N$, we obtain that $S(N) \subseteq N \setminus \{a_0\}$. For the remaining subset relation suppose to the contrary that there is a second element missing from the range of N: there is an element $n_0 \in N$ satisfying $n_0 \notin S(N)$ and $n_0 \neq a_0$. Set $N_0 = N \setminus \{n_0\}$. Note that $a_0 \in N_0$ and that $S(N_0) \subseteq N_0$. Thus $N_0 \in \mathcal{K}$. We also know that $N_0 \subsetneqq N$, yielding a contradiction.

Now let $M \subseteq N$, with $a_0 \in M$, and satisfying $S(M) \subseteq M$. Then $M \in \mathcal{K}$, and thus, again using the minimality of N in $\mathcal{K}$, it follows that $M \supseteq N$. This proves (D3) and completes the proof.

Exercise 1.17
Present the proof of this Theorem.

Recursion and Uniqueness. Before we give a proof of the "essential" uniqueness of the natural numbers, we will follow Dedekind and establish the following general **Recursion Principle**:

Task 1.18
Let A be an arbitrary set, and let $a \in A$ and a function $f : A \to A$ be given. Then there exists a unique map $\varphi : \mathbb{N} \to A$ satisfying

1. $\varphi(0) = a$, and
2. $\varphi \circ S = f \circ \varphi$.

Here is a possible outline for a proof: Consider all subsets $K \subseteq \mathbb{N} \times A$ with the following properties:

1. $(0, a) \in K$, and
2. If $(n, b) \in K$, then $(S(n), f(b)) \in K$.

Clearly $\mathbb{N} \times A$ itself has these properties; we can therefore define the smallest such set: Let

$$L = \bigcap \{K \subseteq \mathbb{N} \times A \mid K \text{ satisfies (1) and (2)}\}.$$

Now show by induction that for every $n \in \mathbb{N}$ there is a unique $b \in A$ with $(n, b) \in L$. This property defines φ by setting $\varphi(n) = b$ for all $n \in \mathbb{N}$.

The Recursion Principle makes it possible to define a recursive procedure (the function φ) via a formula (the function f).

Exercise 1.19
Define addition of an arbitrary natural number n and the fixed natural number m using the Recursion Principle.

Exercise 1.20
Define multiplication of an arbitrary natural number n with the fixed natural number m using the Recursion Principle.

Use the Recursion Principle to show that the set of natural numbers is unique in the following sense:

Task 1.21
Suppose that $\mathbb{N}$, $S : \mathbb{N} \to \mathbb{N}$ and 0 satisfy Axioms (D1)–(D3), and that $\mathbb{N}'$, $S' : \mathbb{N}' \to \mathbb{N}'$ and $0'$ satisfy Axioms (D1)–(D3) as well.

Then there is a bijection[11] $\varphi : \mathbb{N} \to \mathbb{N}'$ such that

1. $\varphi(0) = 0'$, and
2. $\varphi \circ S = S' \circ \varphi$.

[11] A function $f : A \to B$ is a *bijection*, if it is both injective and surjective.