1.4 The Real Numbers

Completeness. While the rational numbers have nice algebraic properties with respect to their addition, their multiplication and their order, they have one crucial deficiency: The set of rational numbers has "holes".

For instance, the increasing sequence of rational numbers

$$1, 1.4, 1.41, 1.414, 1.4142, \ldots$$

approaches the non-rational number $\sqrt{2}$, a fact well known since antiquity.

We want to remedy this deficiency by constructing an ordered field F containing the rational numbers, which is "complete" in the following sense:

> **(C1)** Every increasing bounded sequence of elements in F converges to an element in F.[13]

Calculus books usually introduce completeness of the set of real numbers in this fashion.

It is convenient to describe completeness also in a different way.

We say a **non-empty** set $A \subseteq F$ is *bounded from above*, if there is a $b \in F$ such that $a \leq b$ for all $a \in A$. Such an element b is then called an upper bound for the set A.

If $A \subseteq F$ is bounded from above, we say that A has a *least upper bound*, denoted by $\sup(A) \in F$, if

1. $\sup(A)$ is an upper bound of A, and
2. for all upper bounds b of A, we have $\sup(A) \leq b$.

Note that $\sup(A)$ must be in F, but we do not require that $\sup(A)$ is an element of A.

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[13] A *sequence* is a function $\phi : \mathbb{N} \to F$.

A sequence $\phi : \mathbb{N} \to F$ is called *increasing*, if $m \leq n$ implies $\phi(m) \leq \phi(n)$.

An increasing sequence $\phi : \mathbb{N} \to F$ is called *bounded*, if there is a $b \in F$ such that $\phi(n) \leq b$ for all $n \in \mathbb{N}$.

We say that the increasing sequence ϕ *converges* to $a \in F$, if for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $a - \varepsilon \leq \phi(n) \leq a$ for all $n \geq N$.

Task 1.42
Let $A = \{a \in \mathbb{Q} \mid a^2 < 2\}$. Show that A is bounded from above, but fails to have a least upper bound in $\mathbb{Q}$.

The *greatest lower bound* of a set is defined analogously:

We say a non-empty set $A \subseteq F$ is *bounded from below*, if there is a $b \in F$ such that $b \leq a$ for all $a \in A$. Such an element b is then called a lower bound for the set A.

If $A \subseteq F$ is bounded from below, we say that A has a *greatest lower bound*, denoted by $\inf(A) \in F$, if

1. $\inf(A)$ is a lower bound of A, and
2. for all lower bounds b of A, we have $b \leq \inf(A)$.

Task 1.43
Show the following are equivalent:

1. All subsets of F that are bounded from above have a least upper bound.
2. All subsets of F that are bounded from below have a greatest lower bound.

Completeness can then be stated as follows:

(C2) Every subset A of F, which is bounded from above, has a least upper bound.

Task 1.44
Show that property **(C2)** implies property **(C1)**.

Task 1.45
Show that property **(C1)** implies property **(C2)**.

Constructions of the real numbers. Historically, three "constructions" of the real numbers gained prominence in the 19th century, due to Richard Dedekind (Dedekind cuts), Georg Cantor and Augustin-Louis Cauchy (fundamental sequences), and Paul Bachmann (nested intervals), respectively. We will present the first construction below.

Dedekind Cuts. Given two sets of rational numbers $\emptyset \neq L, U \subseteq \mathbb{Q}$, we say that (L, U) is a *partition* of $\mathbb{Q}$ (into two sets), if $L \cup U = \mathbb{Q}$ and $L \cap U = \emptyset$.

A partition (L, U) of $\mathbb{Q}$ is called a *Dedekind cut*, if the following properties hold:

1. If $a \in L$ and $b \in U$, then $a < b$.
2. U has no minimal element.

Here, the element x of a non-empty set A of rational numbers is called *minimal element* of A, if $x \leq a$ for all $a \in A$.

L and U are complementary sets: $U = \mathbb{Q} \setminus L$, and $L = \mathbb{Q} \setminus U$.

Here are two examples of Dedekind cuts:

Task 1.46
Show that
$$L = \{q \in \mathbb{Q} \mid q \leq -3\},\ U = \{q \in \mathbb{Q} \mid q > -3\}$$
defines a Dedekind cut.

The two sets above "meet" at the rational number -3.

Task 1.47
Show that
$$L = \{q \in \mathbb{Q} \mid q \leq 0 \text{ or } q^2 < 2\},\ U = \{q \in \mathbb{Q} \mid q > 0 \text{ and } q^2 > 2\}$$
defines a Dedekind cut.

Here the two sets of the Dedekind cut "meet" at the irrational number $\sqrt{2}$.

Dedekind then defined the set of real numbers to be the set of all Dedekind cuts:
$$\mathbb{R} = \{(L, U) \mid (L, U) \text{ is a Dedekind cut}\}.$$

Note that the rational number $q \in \mathbb{Q}$ corresponds to the Dedekind cut, defined by $L = (-\infty, q] \cap \mathbb{Q}$, $U = (q, \infty) \cap \mathbb{Q}$. We will denote this Dedekind cut by $\underline{q}$.

Addition of Dedekind cuts. Given two Dedekind cuts (L_1, U_1) and (L_2, U_2) we define their sum to be the Dedekind cut (X, Y), where
$$Y = \{y \in \mathbb{Q} \mid y = u_1 + u_2 \text{ for some } u_1 \in U_1 \text{ and } u_2 \in U_2\},$$
$$\text{and } X = \mathbb{Q} \setminus Y.$$

Task 1.48
Show that (X, Y) is indeed a Dedekind cut.

Task 1.49
Let $p, q \in \mathbb{Q}$. Show: $\underline{p} + \underline{q} = \underline{p+q}$.

Task 1.50
Show that the Dedekind cuts with the addition defined above form an Abelian group (see p. 12). What is the neutral element? What is the additive inverse of a Dedekind cut?

Note that the previous task makes it, in particular, possible to define the difference of two Dedekind cuts.

Next, we can define an order on Dedekind cuts: We say that $(L_1, U_1) \leq (L_2, U_2)$, if $L_1 \subseteq L_2$. In particular, (L, U) is non-negative, if $(-\infty, 0] \cap \mathbb{Q} \subseteq L$. We say $(L_1, U_1) < (L_2, U_2)$, if $(L_1, U_1) \leq (L_2, U_2)$ and $(L_1, U_1) \neq (L_2, U_2)$

Clearly $\leq$ is reflexive, anti-symmetric and transitive (why?). The order is also total:

Task 1.51
For any two Dedekind cuts (L_1, U_1) and (L_2, U_2),

$$(L_1, U_1) \leq (L_2, U_2) \text{ or } (L_2, U_2) \leq (L_1, U_1).$$

It is harder to define the multiplication of Dedekind cuts. If both (L_1, U_1) and (L_2, U_2) are non-negative, we define their product (X, Y) by setting

$$Y = \{y \in \mathbb{Q} \mid y = u_1 \cdot u_2 \text{ for some } u_1 \in U_1 \text{ and } u_2 \in U_2\},$$

$$\text{and } X = \mathbb{Q} \setminus Y.$$

Task 1.52
Check that the product defined above is indeed a Dedekind cut.

To define the product of arbitrary Dedekind cuts, one first needs the following result:

Theorem. Every Dedekind cut is the difference of two non-negative Dedekind cuts.

The product of two arbitrary Dedekind cuts is then defined by "multiplying out"; the concept is well-defined.

With these definitions one can show with quite a bit more work:

Theorem. The real numbers with the addition, multiplication and order defined above form an **ordered Archimedean field**.

The Dedekind cut $\underline{1} := (\mathbb{Q} \cap (-\infty, 1], \mathbb{Q} \cap (1, \infty))$ is the neutral element with respect to multiplication. The existence of a multiplicative inverse is first shown for positive Dedekind cuts, and then generalized to negative Dedekind cuts.

Completeness of Dedekind cuts. Note that a Dedekind cut (L', U') is an upper bound for a set of Dedekind cuts $\mathcal{D}$, if $L \subseteq L'$ for all $(L, U) \in \mathcal{D}$.

Task 1.53
Let
$$\mathcal{D} = \left\{ \left(\mathbb{Q} \cap (-\infty, -\frac{1}{n}], \mathbb{Q} \cap (-\frac{1}{n}, \infty) \right) \mid n \in \mathbb{N} \right\}.$$
Show that $\mathcal{D}$ is bounded from above, then determine its least upper bound.

Finally we can show that the set of real numbers defined via Dedekind cuts is **complete**:

Task 1.54
Show that $\mathbb{R}$, the set of all Dedekind cuts, satisfies Axiom **(C2)**.

Task 1.55
Show that $\mathbb{Q}$ is dense in $\mathbb{R}$: Given two Dedekind cuts $(L_1, U_1) < (L_2, U_2)$, there is a $q \in \mathbb{Q}$ such that
$$(L_1, U_1) \leq \underline{q} \leq (L_2, U_2).$$