2 Numerical Series

From this section onward the standard results from a first Analysis course are a prerequisite. For this section in particular you can (and will need to) use results about numerical sequences.

Given a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers, the *infinite series* $\sum_{n=0}^{\infty} a_n$ is a formal expression of the form

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots$$

The corresponding *sequence of partial sums* $(s_k)_{k\in\mathbb{N}}$ is defined by

$$s_k = a_1 + a_2 + a_3 + \cdots a_k.$$

If the sequence of partial sums converges, with limit s, we say that the series $\sum_{n=0}^{\infty} a_n$ converges, and we write

$$\sum_{n=0}^{\infty} a_n = s.$$

We will often write $\sum a_n$ instead of $\sum_{n=0}^{\infty} a_n$. Sometimes the summation will not start at $n = 0$.

Task 2.1
Show that the series $\sum_{n=0}^{\infty} a_n$ converges if and only if there is a $k \in \mathbb{N}$ such that $\sum_{n=k}^{\infty} a_n$ converges.

This task does not imply that for a given $k \neq 0$

$$\sum_{n=0}^{\infty} a_n = \sum_{n=k}^{\infty} a_n.$$

The following are also direct consequences of the corresponding facts for sequences:

1. If $b \in \mathbb{R}$ and the series $\sum a_n$ converges, then the sum $\sum (b \cdot a_n)$ converges as well, and
$$\sum_{n=0}^{\infty} (b \cdot a_n) = b \cdot \sum_{n=0}^{\infty} a_n.$$
2. If the series $\sum a_n$ and $\sum b_n$ both converge, then their sum $\sum (a_n + b_n)$ converges as well, and
$$\sum_{n=0}^{\infty} (a_n + b_n) = \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n.$$

Task 2.2
If $a_n \geq 0$ for all $n \in \mathbb{N}$, then $\sum a_n$ converges if and only if the corresponding sequence of partial sums (s_k) is bounded.

Task 2.3
Show that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Hint: Show that the partial sums satisfy $s_k \leq 2 - \frac{1}{k}$.

This implies that $\sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2$. Euler showed that the limit is actually equal to $\frac{\pi^2}{6} \approx 1.64493$.

Task 2.4
Show that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (= does not converge).

Hint: Show that the partial sums satisfy $s_{2^k} \geq 1 + \frac{k}{2}$.

Task 2.5
The series $\sum a_n$ converges if and only if for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that whenever $m > n \geq N$ it follows that

$$|a_{n+1} + a_{n+2} + \cdots a_m| < \varepsilon.$$

Task 2.6
If $\sum a_n$ converges, then (a_n) converges to 0.

Note that by the example in Task 2.4 the converse of Task 2.6 does **not** hold.

Task 2.7
Show: If the series $\sum_{n=0}^{\infty} |a_n|$ converges, so does $\sum_{n=0}^{\infty} a_n$.

If $\sum_{n=0}^{\infty} |a_n|$ converges, we say that $\sum_{n=0}^{\infty} a_n$ *converges absolutely*. If on the other hand, $\sum_{n=0}^{\infty} a_n$ converges while $\sum_{n=0}^{\infty} |a_n|$ diverges, we say that $\sum_{n=0}^{\infty} a_n$ *converges conditionally*.

Task 2.8
Show that the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

converges conditionally.

The example above is a special case of the next task:

Task 2.9
Suppose the sequence (a_n) satisfies

1. $a_0 \geq a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$, and
2. the sequence (a_n) converges to 0,

then $\displaystyle\sum_{n=0}^{\infty}(-1)^n a_n$ converges.

Given a series $\displaystyle\sum_{n=0}^{\infty} a_n$, we say the series $\displaystyle\sum_{n=0}^{\infty} b_n$ is a *rearrangement* of $\displaystyle\sum_{n=0}^{\infty} a_n$, if there is a bijection $\varphi : \mathbb{N} \to \mathbb{N}$ such that $b_{\varphi(n)} = a_n$ for all $n \in \mathbb{N}$.

Task 2.10
If the series $\displaystyle\sum_{n=0}^{\infty} a_n$ converges absolutely, then any rearrangement of $\displaystyle\sum_{n=0}^{\infty} a_n$ converges to the same limit.

In other words: If a series is absolutely convergent, then it is "infinitely commutative." If, on the other hand, a series converges only conditionally, then commutativity fails in a spectacular way:

Task 2.11
Suppose that the series $\sum_{n=0}^{\infty} a_n$ converges conditionally. Then for every $s \in \mathbb{R}$, there is a rearrangement $\sum_{n=0}^{\infty} b_n$ of $\sum_{n=0}^{\infty} a_n$ such that $\sum_{n=0}^{\infty} b_n$ converges to s.

Here are two hints to get you started on this problem:

1. Let $a_n^+ = \max\{a_n, 0\}$ and $a_n^- = \max\{-a_n, 0\}$. Thus $a_n = a_n^+ - a_n^-$ and $|a_n| = a_n^+ + a_n^-$. Observe that both series $\sum_{n=0}^{\infty} a_n^+$ and $\sum_{n=0}^{\infty} a_n^-$ do not converge. Therefore both partial sums are not bounded.
2. The series in Task 2.8 actually converges to $\ln 2 \approx 0.693147$. Can you find a recipe how to rearrange the series so that the rearrangement converges to 1 instead?

3 Sequences and Series of Functions

3.1 Pointwise and Uniform Convergence

We now turn our attention to the convergence of sequences and series of functions. Here is the natural definition to extend the notion of convergence from numbers to functions:

Let $D \subseteq \mathbb{R}$. Given functions $f_n : D \to \mathbb{R}$ and $f : D \to \mathbb{R}$, we say the sequence $(f_n)_{n\in\mathbb{N}}$ *converges to f pointwise* if $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in D$.

Equivalently, this means that for all $x \in D$ and for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$ it follows that $|f_n(x) - f(x)| < \varepsilon$.

Task 3.1
Let $f_n : [0,1] \to \mathbb{R}$ be given by $f(x) = x^n$. Find a suitable $f : [0,1] \to \mathbb{R}$ such that (f_n) converges to f pointwise.

This example reveals the first deficiency of pointwise convergence: the pointwise limit of a sequence of continuous functions is not necessarily continuous.

The next example shows that the pointwise limit of a sequence of bounded functions is not necessarily bounded. We say a function $f : D \to \mathbb{R}$ is *bounded* if there is an $M > 0$ such that $|f(x)| \leq M$ for all $x \in D$.

Task 3.2
Let $f_n : (-1,1) \to \mathbb{R}$ be given by $f_n(x) = \sum_{k=0}^{n} x^k$. Show that (f_n) converges pointwise to the function $f(x) = \dfrac{1}{1-x}$.

Let us say that a sequence $f_n : D \to \mathbb{R}$ is *uniformly bounded*, if there is an $M > 0$ such that $|f_n(x)| \leq M$ for all $x \in D$ and $n \in \mathbb{N}$.

Assuming this extra assumption, we obtain a positive result:

Task 3.3
Suppose the sequence $f_n : D \to \mathbb{R}$ is uniformly bounded and converges pointwise to the function f. Then f is bounded.

Pointwise convergence also does not interact nicely with Riemann integration:

Task 3.4
Let $f_n : [0, 1] \to \mathbb{R}$ be given by

$$f_n(x) = \begin{cases} 0, & \text{if } x = 0 \\ n, & \text{if } 0 < x \leq \frac{1}{n} \\ 0, & \text{if } \frac{1}{n} < x \leq 1 \end{cases}$$

Show that this sequence converges pointwise to the zero-function.

Observe that $\displaystyle\int_0^1 f_n(x)\, dx = 1$ for all n, while the pointwise limit has integral 0.

We have already seen in Task 3.1 that the pointwise limit of differentiable functions is not necessarily differentiable. The next example shows that even in the case when the pointwise limit is differentiable, its derivative does not necessarily have the desired properties.

Task 3.5
Let $f_n : [-1, 1] \to \mathbb{R}$ be defined by $f_n(x) = \dfrac{x}{1 + nx^2}$. Show that each f_n is differentiable, that (f_n) has as its pointwise limit f the zero function, but that $\lim_{n\to\infty} f_n'(0) \neq f'(0)$.

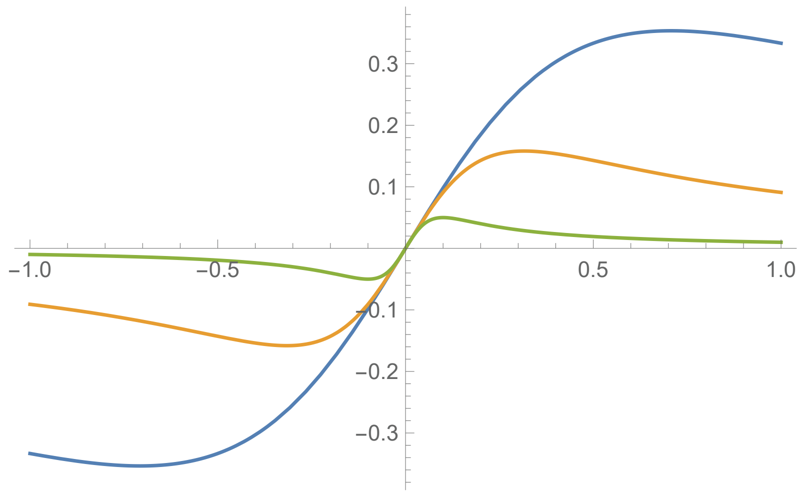

Figure 1: The functions f_2, f_{10} and f_{100} from Task 3.5

All these examples show that pointwise convergence is not such a useful property[14].

We will therefore study a different limit concept for functions:

Let $D \subseteq \mathbb{R}$. Given functions $f_n : D \to \mathbb{R}$ and $f : D \to \mathbb{R}$, we say the sequence $(f_n)_{n\in\mathbb{N}}$ *converges to f uniformly* if for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$ and for all $x \in D$ it follows that $|f_n(x) - f(x)| < \varepsilon$.

Task 3.6
Let functions $f_n : D \to \mathbb{R}$ and $f : D \to \mathbb{R}$ be given. If (f_n) converges to f uniformly, then (f_n) converges to f pointwise.

Task 3.7
Show that the converse of Task 3.6 is false.

[14] Actually, in the case of integration, this led to the development of a different notion of integration: the Lebesgue integral with its Dominated Convergence Theorem.

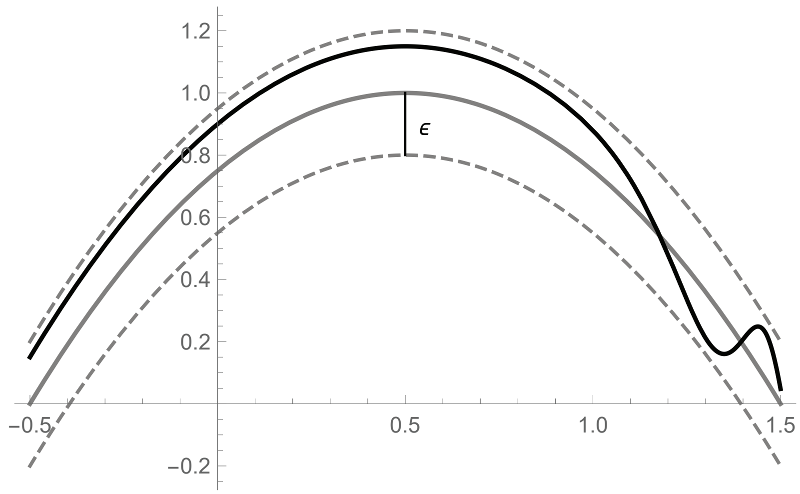

Figure 2: Uniform convergence: The function f_n (in black) lies in an ε-tube around the function f (in gray).

Task 3.8
Let a sequence of functions $f_n : D \to \mathbb{R}$ be given. The sequence (f_n) converges uniformly if and only if for all $\varepsilon > 0$ there is an $n \in \mathbb{N}$ such that for all $x \in D$ and for all $m, n \geq N$ it follows that $|f_m(x) - f_n(x)| < \varepsilon$.

Task 3.9
Let a sequence of functions $f_n : D \to \mathbb{R}$ and numbers $M_n \geq 0$ be given. Suppose

$$|f_n(x)| \leq M_n \text{ for all } x \in D \text{ and } n \in \mathbb{N}.$$

Show: If $\sum M_n$ converges, then $\sum f_n$ converges uniformly (and absolutely).

The next two results highlight some of the permanence properties of uniform convergence:

Task 3.10
Let $f_n : D \to \mathbb{R}$ be continuous functions such that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to some function f. Then f is continuous.

Task 3.11
Let $f_n : D \to \mathbb{R}$ be bounded functions such that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to some function f. Then f is bounded.

There are two more permanence results for uniform convergence:

Theorem 3.1. Let $f_n : [a, b] \to \mathbb{R}$ be Riemann-integrable functions such that (f_n) converges uniformly to some function f. For $t \in [a, b]$, let $F_n(t) = \int_a^t f_n(x)\,dx$.

Then f is Riemann integrable, and moreover (F_n) converges uniformly to the function F, defined by $F(t) = \int_a^t f(x)\,dx$.

Theorem 3.2. Let $f_n : [a, b] \to \mathbb{R}$ be differentiable functions such that (f_n') converges uniformly to some function g. Assume additionally that for some $x_0 \in [a, b]$ the sequence $(f_n(x_0))$ converges.

Then (f_n) converges uniformly to some function f, f is differentiable on $[a, b]$, and $f'(x) = g(x)$ for all $x \in [a, b]$.