Math 3341 The Euler-Mascheroni Constant

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In an Analysis course, the logarithm is usually defined for $x > 0$ as

$$\ln(x) = \int_1^x \frac{dt}{t}.$$

For $x \geq 1$, $\ln(x)$ is thus the area enclosed by the graph of the function $f(t) = \frac{1}{t}$, the t-axis, and the lines $t = 1$ and $t = x$.

For $n \in \mathbb{N}$, we let

$$c_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n.$$

1. Show that (c_n) is a decreasing sequence that is bounded from below. Its limit, approximately 0.577, is called the *Euler-Mascheroni Constant* and usually denoted by γ.

2. Set
$$b_n = 1 - \frac{1}{2} + \frac{1}{3} - \cdots \pm \frac{1}{n}.$$
Show that (b_n) converges to $\ln 2$. Hint: $b_{2n} = c_{2n} - c_n + \ln 2$.

3. Show that
$$\gamma = \lim_{n\to\infty} (1 + \frac{1}{2} + \cdots + \frac{1}{n} - \frac{1}{n-1} - \frac{1}{n+2} - \cdots - \frac{1}{n^2}).$$
Hint: (c_{n^2}) converges to γ.