Multi-Resolution Analysis for the Haar Wavelet

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1 The space $L^2([0,1))$ and its scalar product

We will denote by $L^2([0,1))$ the vector space of all functions $f:[0,1)\to\mathbb{R}$ satisfying[1]

$$\int_0^1 |f(x)|^2\,dx < \infty. \tag{1}$$

On $L^2([0,1))$ one can define a scalar product as follows:

$$< f,g >= \int_0^1 f(x)\cdot g(x)\,dx. \tag{2}$$

Similarly to the case of $\mathbb{R}^n$, the scalar product automatically defines a norm on $L^2([0,1))$ via the definition

$$\|f\| = \sqrt{< f,f >} = \sqrt{\int_0^1 |f(x)|^2\,dx} \tag{3}$$

Finally, we say that a sequence (f_n) of functions in $L^2([0,1))$ converges to a function $f(x)\in L^2([0,1))$ in the L^2-sense, if

$$\lim_{n\to\infty} \|f_n - f\| = 0 \tag{4}$$

2 Orthonormal sets

We say that a set B of elements in $L^2([0,1))$ is an orthonormal set, if the scalar product of each element in B with itself equals 1, and the scalar product of two different elements in B is equal to 0:

1. $< f,f >= 1 \quad$ for all $f\in B$
2. $< f,g >= 0 \quad$ for all $f,g\in B$ satisfying $f\neq g$.

[1] More precisely: those (equivalence classes of) measurable functions on $[0,1)$ whose square is Lebesgue-integrable. Since we will be ultimately interested only in the discrete case anyway, you can just think of bounded piecewise-continuous functions with the Riemann integral.

An orthonormal set B is automatically linearly independent.[2] Our **ultimate goal** will be to find a particular orthonormal set $B = \{f_1(x), f_2(x), \ldots\}$ such that we can approximate every function $f(x) \in L^2([0,1))$ by linear combinations of the elements in B; more precisely, given $f(x) \in L^2([0,1))$, we will be able to find scalars (a_k) such that the sequence

$$\left(\sum_{k=1}^{n} a_k f_k(x) \right) \tag{5}$$

converges to $f(x)$ in the L^2-sense.[3]

3 The Haar scaling function

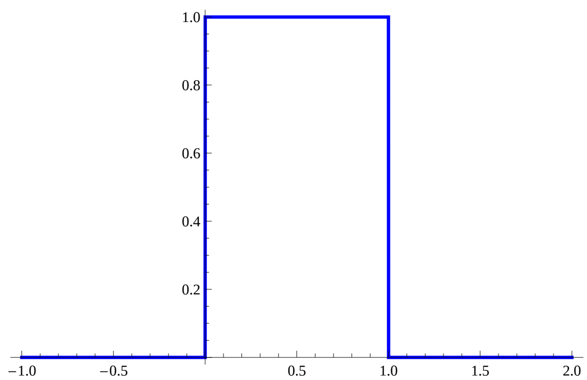

Figure 1: The Haar scaling function $\phi(x)$

[2]Indeed assume that for some real numbers $a_1, a_2, \ldots, a_n$ and some distinct elements $g_1, g_2, \ldots, g_n$ in B,

$$a_1 g_1(x) + a_2 g_2(x) + \cdots + a_n g_n(x) = 0 \text{ for all } x.$$

Then for any k with $k \in \{1, \ldots, n\}$

$$< a_1 g_1 + a_2 g_2 + \cdots + a_n g_n, g_k >=< 0, g_k >= 0.$$

On the other hand,

$$< a_1 g_1 + a_2 g_2 + \cdots + a_n g_n, g_k >= a_1 < g_1, g_k > + a_2 < g_2, g_k > + \cdots + a_n < g_n, g_k >= a_k < g_k, g_k >= a_k.$$

So $a_k = 0$ for all k.

[3]We basically already know one example of such a set: It is known that the set

$$F = \{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(x), \frac{1}{\sqrt{\pi}}\cos(2x), \ldots, \frac{1}{\sqrt{\pi}}\sin(x), \frac{1}{\sqrt{\pi}}\sin(2x), \ldots\}$$

forms an orthonormal set with which we can approximate all elements in $L^2([-\pi, \pi])$ in this fashion.

We denote by $\phi(x)$ the following function:

$$\phi(x) = \begin{cases} 1, & \text{if } x \in [0,1) \\ 0, & \text{if } x < 0 \text{ or } x \geqslant 1 \end{cases} \tag{6}$$

$\phi : \mathbb{R} \to \mathbb{R}$ is called the HAAR SCALING FUNCTION, or the Haar "father" wavelet. Throughout we will identify $\phi(x)$ with its restriction to $[0,1)$.

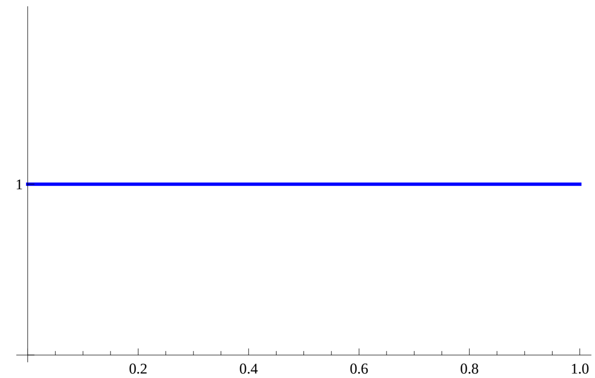

Figure 2: The Haar scaling function $\phi(x)$ restricted to $[0,1)$

Let V_0 denote the one-dimensional vector space spanned by $\phi(x)$; this is nothing else but the set of all functions constant on $[0,1)$ (and vanishing elsewhere).

Next we consider the functions $2^{1/2}\phi(2x)$ and $2^{1/2}\phi(2x-1)$. They span a two-dimensional vector space, denoted by V_1, consisting of all functions on $[0,1)$ that are constant both on $[0,\frac{1}{2})$ and on $[\frac{1}{2},1)$.

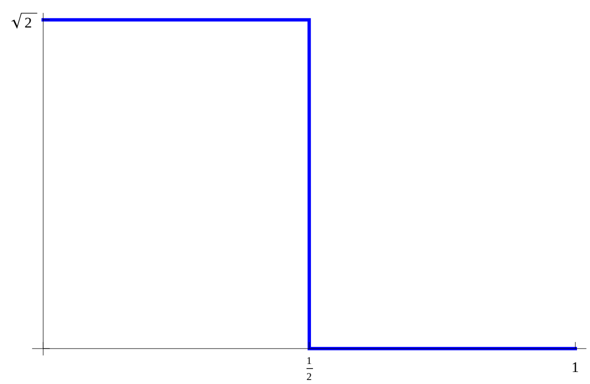

Figure 3: The function $2^{1/2}\phi(2x)$

Note that $V_0 \subset V_1$. Indeed the function $\phi(x)$ is a linear combination of $2^{1/2}\phi(2x)$ and $2^{1/2}\phi(2x-1)$:

$$\phi(x) = \frac{1}{\sqrt{2}}\left(2^{1/2}\phi(2x)\right) + \frac{1}{\sqrt{2}}\left(2^{1/2}\phi(2x-1)\right) \tag{7}$$

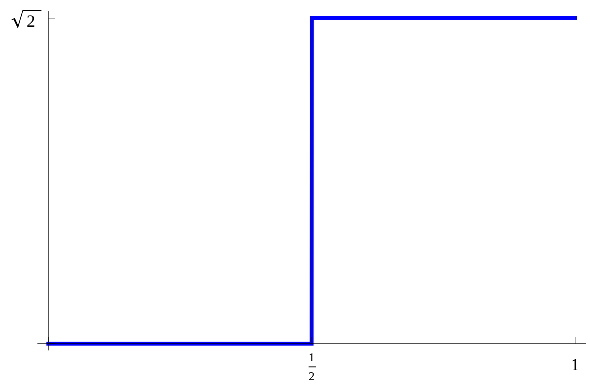

Figure 4: The function $2^{1/2}\phi(2x-1)$

More generally, an equation of the form

$$\phi(x) = \sum_{k=0}^{n} a_k \phi(2x-k) \tag{8}$$

is called a **dilation equation**. Functions ϕ that satisfy a dilation equation are rare and hard to find.

Continuing in this fashion, we can define a 2^j-dimensional vector space V_j, spanned by the functions

$$2^{j/2}\phi(2^j x), 2^{j/2}\phi(2^j x-1), \ldots, 2^{j/2}\phi(2^j x-(2^j-1)).$$

The vector space V_j consists of all functions on $[0,1)$ that are constant on intervals of the form $[k2^{-j},(k+1)2^{-j})$ for $k=0,1,2,\ldots 2^j-1$. Figure 5 shows the function $2^{3/2}\phi(2^3x-5)$ contained in V_3. We have $V_0 \subset V_1 \subset \cdots \subset V_j \subset \cdots$.

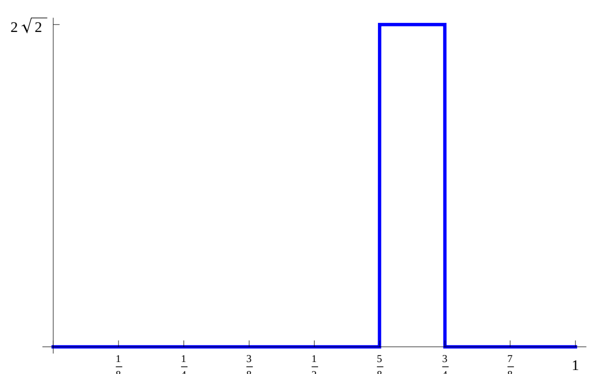

Figure 5: The function $2^{3/2}\phi(2^3x-5)$

You should have wondered by now why the factor $2^{j/2}$ is included. The answer is straightforward: this way the functions form an orthonormal set!

Exercise 1
Show that the set $\{2^{j/2}\phi(2^j x), 2^{j/2}\phi(2^j x-1), \ldots, 2^{j/2}\phi(2^j x-(2^j-1))\}$ forms an orthonormal set of functions in the vector space V_j.

4 Using V_j to approximate functions in $L^2([0,1))$

A function $f \in V_j$ has the form

$$f(x) = a_0 2^{j/2}\phi(2^j x) + a_1 2^{j/2}\phi(2^j x - 1) + \cdots + a_{2^j-1} 2^{j/2}\phi(2^j x - (2^j-1)). \tag{9}$$

Since the functions on the right side form an orthonormal set, the coefficients a_k are given by the formula

$$a_k = < f(x), 2^{j/2}\phi(2^j x - k) > = \int_0^1 f(x) \cdot 2^{j/2}\phi(2^j x - k)\, dx \tag{10}$$

Exercise 2
Take the scalar product with $2^{j/2}\phi(2^j x - k)$ on both sides of (9) to verify Formula (10).

The same formula for the coefficients can be used to approximate functions in $L^2([0,1))$ by a function in V_j. Let $f(x)$ be a function in $L^2([0,1))$, and set

$$f_j(x) = a_0 2^{j/2}\phi(2^j x) + a_1 2^{j/2}\phi(2^j x - 1) + \cdots + a_{2^j-1} 2^{j/2}\phi(2^j x - (2^j-1)), \tag{11}$$

where the coefficients a_k are computed via Formula (10).

Alfred Haar (1885–1933) showed in 1910 that, if $f(x)$ is continuous, the sequence $(f_j(x))$ converges to $f(x)$ uniformly. If, on the other hand, $f(x) \in L^2([0,1))$, then

$$\lim_{j\to\infty} \|f - f_j\| = 0.$$

Figures 6 and 7 show the approximation of a function (dashed line) by an element in V_4 and V_7, respectively (solid line).

While we have found nice orthonormal bases for all the vector spaces V_j, we still fall short of our goal: If we take two basis elements from different V_j's, their scalar product will not necessarily equal zero, because the intervals where the basis elements are equal to 1 may overlap.

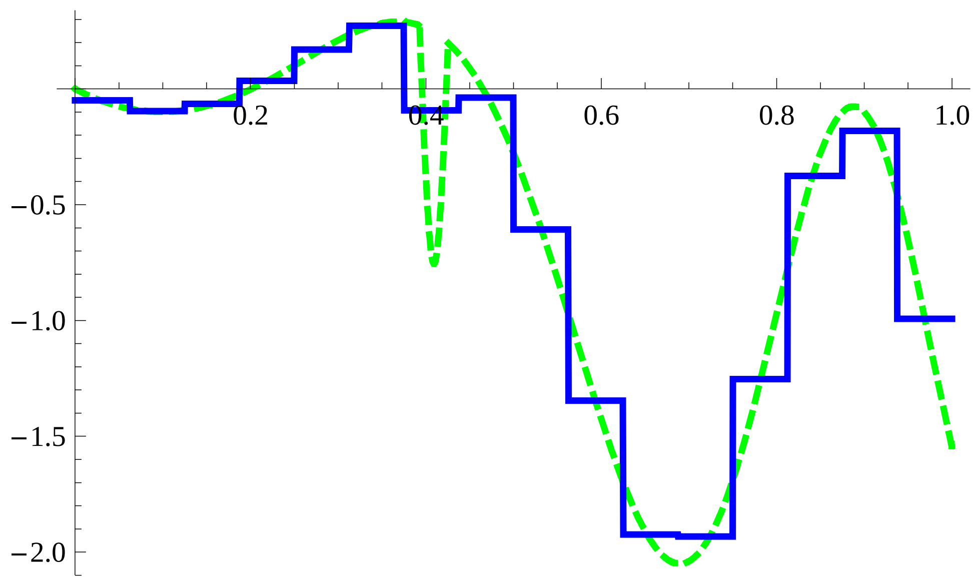

Figure 6: Approximating a function by an element in V_4

5 The Haar wavelet

Let's see whether we can remedy this deficiency step by step. We want to find a function $\psi(x)$ in V_1, such that the linear combinations of $\phi(x)$ and $\psi(x)$ span the vector space V_1, and such that the following conditions are satisfied:

1. $< \phi, \psi >= 0$
2. $< \psi, \psi >= 1$

Since $\psi \in V_1$ we can find scalars a_1 and a_2 such that

$$\psi(x) = a_1\sqrt{2}\phi(2x) + a_2\sqrt{2}\phi(2x-1). \tag{12}$$

Note also that

$$\phi(x) = \phi(2x) + \phi(2x-1). \tag{13}$$

Using (12) and (13), the first condition becomes

$$< \phi, \psi >=< \phi(2x) + \phi(2x-1), a_1\sqrt{2}\phi(2x) + a_2\sqrt{2}\phi(2x-1) >= a_1/\sqrt{2} + a_2/\sqrt{2} = 0, \tag{14}$$

so $a_2 = -a_1$. The second condition yields:

$$< \psi, \psi >=< a_1\sqrt{2}\phi(2x)+a_2\sqrt{2}\phi(2x-1), a_1\sqrt{2}\phi(2x)+a_2\sqrt{2}\phi(2x-1) >= a_1^2+a_2^2 = 1, \tag{15}$$

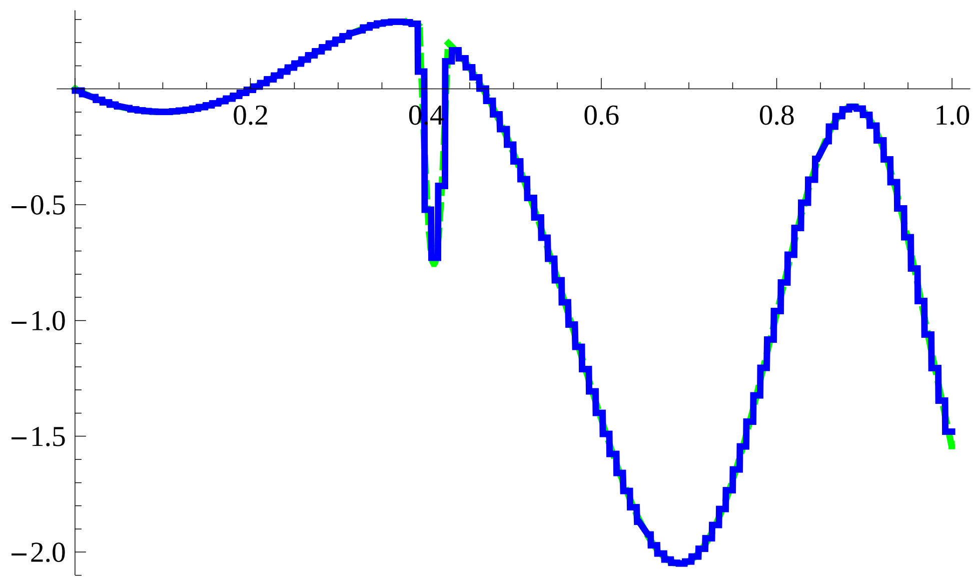

Figure 7: Approximating a function by an element in V_7

Solving (14) and (15) for a_1 and a_2, we obtain[4]

$$\psi(x) = \phi(2x) - \phi(2x-1) = \begin{cases} 1, & \text{if } x \in [0, \frac{1}{2}) \\ -1, & \text{if } x \in [\frac{1}{2}, 1) \\ 0, & \text{otherwise} \end{cases} \tag{16}$$

The function $\psi(x)$ is called the HAAR "MOTHER" WAVELET; its graph is depicted in Fig-

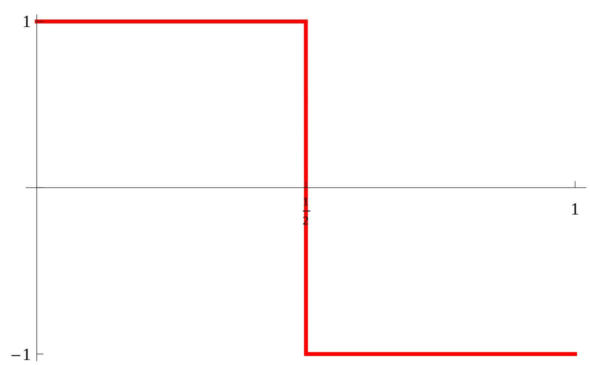

Figure 8: The Haar mother wavelet $\psi(x)$

ure 8.

[4]There are actually two solutions; it suffices for us to consider the solution for which $a_1 > 0$. Why?

We will denote the vector space spanned by the function $\psi(x)$ as W_0. It is customary to write

$$V_1 = V_0 \oplus W_0. \tag{17}$$

Here the symbol $\oplus$ is used to indicate that each element in V_1 can be written in a unique way as the sum of an element in V_0 and an element in W_0 and that the scalar product of any element in V_0 with any element in W_0 equals zero.

Exercise 3

Show that the functions $\sqrt{2}\psi(2x)$ and $\sqrt{2}\psi(2x-1)$ are elements in V_2.

Exercise 4

Show that the set $\{\sqrt{2}\psi(2x),\ \sqrt{2}\psi(2x-1)\}$ forms an orthonormal set.

Exercise 5

Show that $< \sqrt{2}\psi(2x), f(x) >= 0$ for all functions $f(x) \in V_1$. (The same result holds for $\sqrt{2}\psi(2x-1)$.)

These two functions are shown in Figures 9 and 10, respectively.

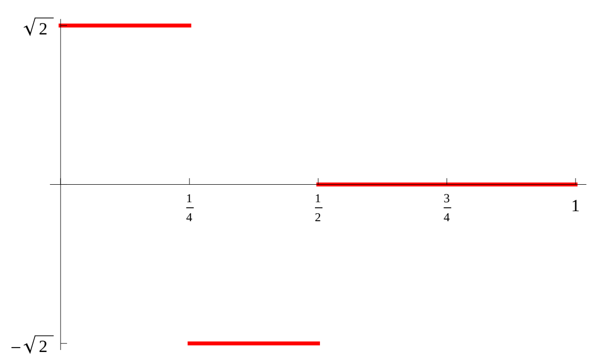

Figure 9: The function $\sqrt{2}\,\psi(2x)$ in W_1

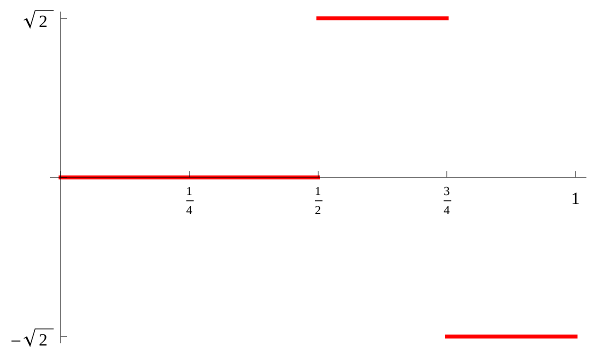

Figure 10: The function $\sqrt{2}\,\psi(2x-1)$ in W_1

Let's denote the vector space spanned by $\sqrt{2}\psi(2x)$ and $\sqrt{2}\psi(2x-1)$ as W_1. The three exercises above show that

$$V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1, \tag{18}$$

meaning once again that each element in V_2 can be written in a unique way as the sum of an element in V_1 and an element in W_1 and that the scalar product of any element in V_1 with any element in W_1 equals zero.

Continuing in this fashion, we can write

$$V_{j+1} = V_j \oplus W_j, \tag{19}$$

where W_j is the vector space spanned by the functions

$$2^{j/2}\psi(2^j x), 2^{j/2}\psi(2^j x - 1), \ldots, 2^{j/2}\psi(2^j x - (2^j - 1)).$$

The formula

$$V_j = V_0 \oplus W_0 \oplus W_1 \oplus \cdots \oplus W_{j-1}$$

is called Multi-Resolution Analysis.

6 A discrete example

A function in V_4 is determined by its 16 coefficients. Suppose the vector of coefficients is

$$(a_0, a_1, a_2, \ldots, a_{15}) = (180, 167, 244, 190, 159, 242, 176, 192, 168, 250, 175, 219, 193, 232, 200, 234) \tag{20}$$

The corresponding function

$$f(x) = \sum_{n=0}^{15} a_n \; 4\phi(16x - n) \tag{21}$$

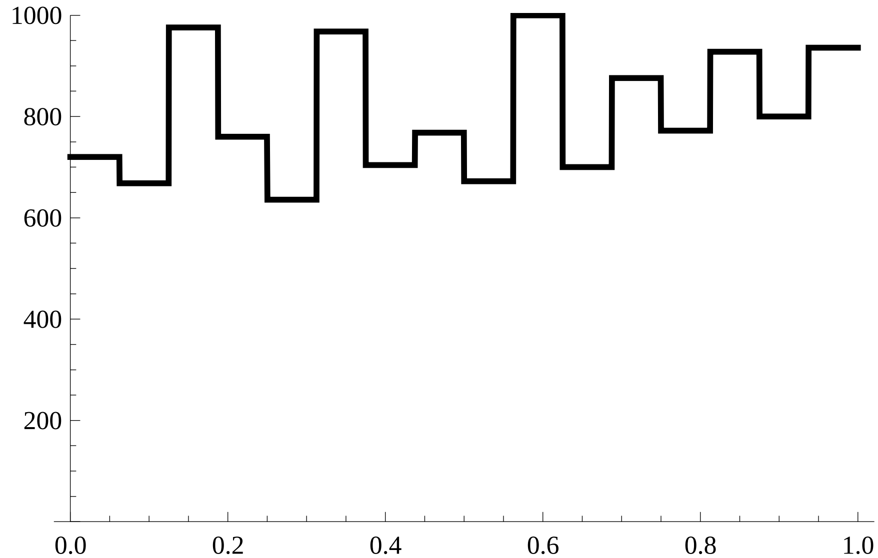

Figure 11: A function in V_4

is shown in Figure 11. How can we write this function f as a sum of a function in V_3 and a function in W_3? Let's start with the component g in V_3. By the discussion following the derivation of Formula (10), the function g is of the form

$$g(x) = \sum_{k=0}^{7} b_k \; 2^{3/2}\phi(8x-k), \tag{22}$$

with the coefficients b_k computed as $b_k =< f(x), 2^{3/2}\phi(8x-k) >$. Thus we obtain

$$\begin{aligned}
b_k &= < f(x), 2^{3/2}\phi(8x-k) > \\
&= < \sum_{n=0}^{15} a_n \; 4\phi(16x-n), 2^{3/2}\phi(8x-k) > \\
&= \sum_{n=0}^{15} a_n \; 8\sqrt{2} < \phi(16x-n), \phi(8x-k) > \\
&= a_{2k} \; 8\sqrt{2} < \phi(16x-2k), \phi(8x-k) > \; + a_{2k+1} \; 8\sqrt{2} < \phi(16x-(2k+1)), \phi(8x-k) > \\
&= a_{2k} \; 8\sqrt{2}\frac{1}{16} + a_{2k+1} \; 8\sqrt{2}\frac{1}{16}
\end{aligned}$$

Here we are using the fact that $< \phi(16x-n), \phi(8x-k) > \neq 0$ only if $n = 2k$ or $n = 2k+1$. (Draw a picture). More specifically, in these two cases $< \phi(16x-n), \phi(8x-k) >= \frac{1}{16}$.

Simplifying the last expression we obtain

$$b_k = \frac{a_{2k} + a_{2k+1}}{\sqrt{2}} \tag{23}$$

This means that we can compute the vector representing the function in V_3 by multiplying the vector in (20) by the 16×8 matrix

$$\begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}}
\end{pmatrix} \tag{24}$$

on the right. In our particular case, the vector (rounded to the nearest integer) representing the function in V_3 is given by

$$(245, 307, 284, 260, 296, 279, 301, 307). \tag{25}$$

Figure 12 shows the function in V_4 and its "blurry" counterpart in V_3.

Exercise 6

Let $f = g + h$, with $f \in V_4$, $g \in V_3$, and $h \in W_3$. Show that if

$$f(x) = \sum_{n=0}^{15} a_n \; 4\phi(16x - n)$$

$$\text{and } h(x) = \sum_{k=0}^{7} c_k \; 2^{3/2}\psi(8x - k),$$

then

$$c_k = \frac{a_{2k} - a_{2k+1}}{\sqrt{2}}, \tag{26}$$

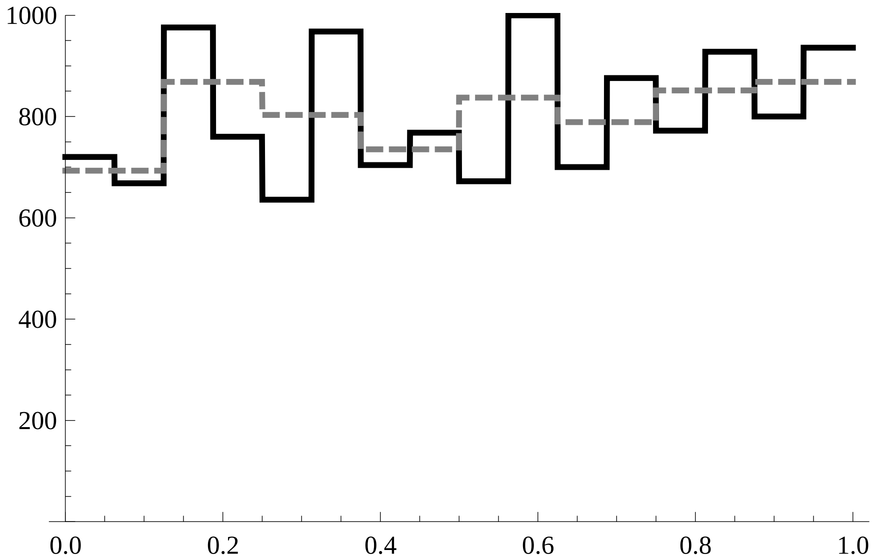

Figure 12: A function in V_4 (black) and its "orthogonal projection" onto V_3 (dashed)

You can either use the fact that $h = f - g$, or repeat the computation on p. 10.

In other words, this time we obtain the vector representing the function in W_3 by multiplying the vector in (20) by the matrix

$$\begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}}
\end{pmatrix} \tag{27}$$

In our particular case, the vector (rounded to the nearest integer) representing the function in W_3 is given by

$$(9, 38, -59, -11, -58, -31, -28, -24). \tag{28}$$

Figure 13 shows this as a function in W_3.

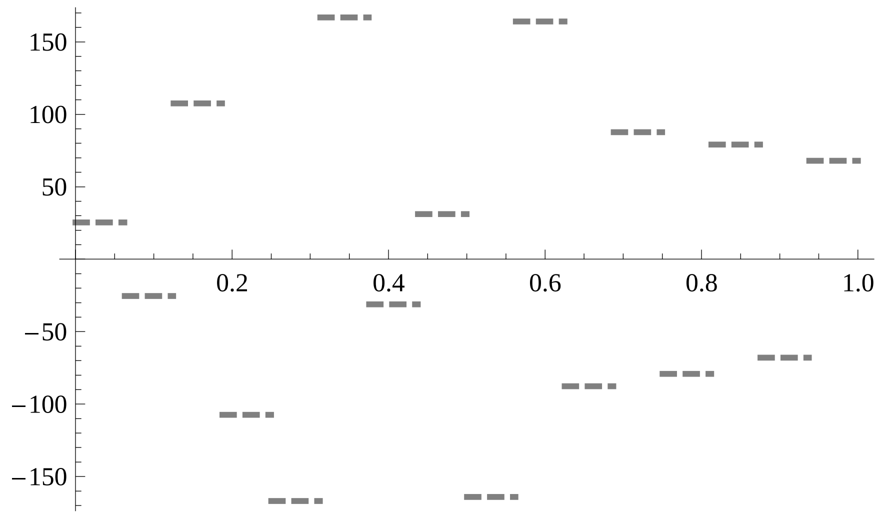

Figure 13: The "orthogonal projection" of our function onto W_3 (dashed)

If we are "joining" the vectors in (25) and (28), we obtain

$$(245, 307, 284, 260, 296, 279, 301, 307, 9, 38, -59, -11, -58, -31, -28, -24). \tag{29}$$

Since our function in V_4 is the sum of its orthogonal projections onto V_3 and W_3, we will be able to retrieve the vector in (20) from the vector (29). The cumulative energy plots of both vectors are shown in Figure 14, indicating that the vector in (29) has a higher energy concentration than the original vector (20) and thus may be considered as a compressed version of the vector in (20).

Exercise 7

This section has shown how to compute vector (29) from vector (20). Can we reverse the procedure? Suppose our procedure produces as the vector in (29)

$$(200, 350, 351, 130, 115, 215, 122, 308, 15, 35, 47, 23, -12, -32, 67, -23).$$

What does the corresponding original vector (20) look like?

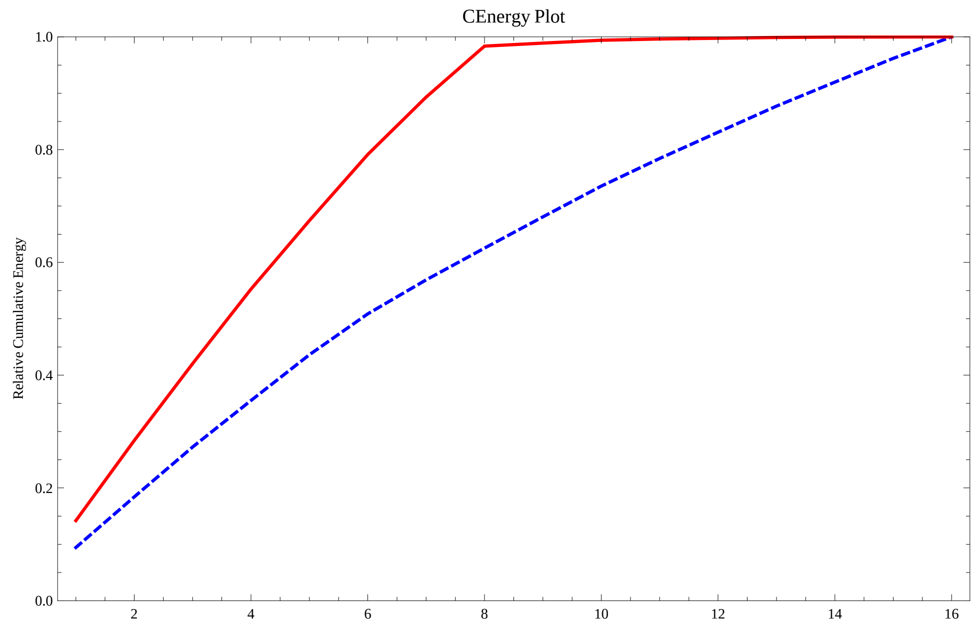

Figure 14: The original vector (20) is shown dashed, while vector (29) is depicted solid.

7 Concluding Remarks

We have outlined a general procedure: (1) start with a father wavelet $\phi(x)$, (2) use a dilation equation to construct an increasing sequence of vector spaces

$$V_1 \subset V_2 \subset \cdots \subset V_j \subset \cdots ,$$

capable of approximating functions in $L^2([0,1))$, (3) construct the corresponding mother wavelet, and (4) ultimately produce a multi-resolution analysis

$$V_j = V_0 \oplus W_0 \oplus W_1 \oplus \cdots \oplus W_{j-1}$$

Our choice for $\phi(x)$ was the constant 1, with the dilation equation

$$\phi(x) = \phi(2x) + \phi(2x-1).$$

As you will see later in the course, in the 1980's other possible candidates emerged.

Acknowledgment. This exposition is based on material in *A First Course in Wavelets with Fourier Analysis* by Albert Boggess & Francis J. Narcowich.