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2.19

Moderator: helmut [Moderator Control Panel]

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Author	Message
helmut	**Post subject:** 2.19 **Posted:** Sun, 05 Apr 2020 17:02

online

Site Admin

Joined: Sat, 26 Apr 2003 15:14
Posts: 2259
Location: El Paso TX (USA)

Here s an outline for the missing direction, using the last hint in the notes.

Let $B=\{b\in\mathbb{R} \mid b \text{ is an upper bound of A}\}$

Step 1:
Show that B is an interval of the form (b,∞) or of the form $[b,\infty)$ for some $b\in\mathbb{R}$.

Step 2:
Show $B=[b,\infty)$. How: It suffices to show that the limit of any decreasing sequence of elements in B converges to an element in B. (Why?) You may use the CA for decreasing sequences.

Step 3:
Show that b is the least upper bound of A.

____
The greater danger for most of us lies not in setting our aim too high and falling short; but in setting our aim too low, and achieving our mark. - Michelangelo Buonarroti

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Author	Message
solivas11	**Post subject:** Re: 2.19 **Posted:** Tue, 14 Apr 2020 00:03

offline

Member

Joined: Mon, 30 Mar 2020 21:12
Posts: 15

2.19 The Completeness Axiom is equivalent to the following: Every non-empty set ofreal numbers which is bounded from above has a supremum.
$(CA \iff \quad supremum\ property)$

First half (Vivian's portion - $sup\ property \Rightarrow CA$)

Suppose every set bounded from above has a supremum.
Let a_n be an increasing, bounded sequence.
Let $A=\{a_n|n\in\mathbb{N}\},\ a=sup(A)$

$\forall \varepsilon > 0 \ \exists a_m \ s.t. \ |a - a_m| < \varepsilon$
$a - \varepsilon < a_m \ < \ a + \varepsilon$
$\Rightarrow$
$a - \varepsilon < \ a_m \ < \ a$
since a_n bounded from above.

Since (a_n) is an increasing sequence,
if $m < n, \ a_m < a_n,$
then $a - \varepsilon < \ a_m < a_n \ < \ a,$
so $|a_n - a| < \varepsilon$
Thus $a_n \rightarrow \ a$

Other half ($CA \Rightarrow \ sup \ property$)

Let a_n be an increasing, bounded sequence.
Let $A = \{a_n | n \in \mathbb{N}\}$
Let $B = \{b \in \mathbb{R} | \ b = UB(A) \ is \ true\}$
$\iff \ B = \{b \in \mathbb{R} | \ \forall \ a \in A : \ a \leq b\}$

> **helmut wrote:**
>
> Step 1:
> Show that B is an interval of the form (b, ∞) or of the form $[b, \infty)$ for some $b \in \mathbb{R}$.

$b = UB(A) \iff \exists b \in \mathbb{R}, \ \forall \ a \in A : \ a \leq b$
Let $\delta \in \mathbb{R} : \ \delta > 0$
Since $a \leq \ b < b + \delta \Rightarrow a < b + \delta \ \forall \delta$
Then, $b + \delta = UB(A) \ \forall \delta$
By this, B is unbounded from above.
If $a < b + \delta \ \forall \delta, \ then \ B = (b, \infty)$
For a special case $\delta \geq 0$, if $a < b + \delta \ \forall \delta, \ then \ B = [b, \infty)$

> **helmut wrote:**
>
> Step 2:
> Show $B = [b, \infty)$. How: It suffices to show that the limit of any decreasing sequence of elements in B converges to an element in B. (Why?) You may use the CA for decreasing sequences.

Let b_n be a decreasing sequence whose range is defined as
$\{b \in \mathbb{R} | \ b = UB(A) \ is \ true\}$
$\iff$
b_n be a decreasing sequence whose range is defined by B
Since B is bounded from below by $[b, \infty)$, and b_n is decreasing, by $CA, \ b_n \rightarrow \ k \in \mathbb{R}$

Now do a proof by contradiction to show $k \in [b, \infty)$
We know $b_n \in [b, \infty)$
Suppose $b_n \rightarrow \ k, \ where \ k < b$
Let $\varepsilon = \dfrac{b - k}{2}$

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } |b_n - k| < \varepsilon \ \forall n \geq N$

$|b_n - k| < \frac{b-k}{2}$

$\frac{k-b}{2} + k < b_n < \frac{b-k}{2} + k$

$\frac{k-b}{2} + k < b_n < \frac{k-b}{2} + b$

$k < b_n + \frac{b-k}{2} < b$

$\Rightarrow$

$b_n + \varepsilon < b$

But we previously stated that $b_n \in [b, \infty)$
Yet, we showed $b_n < b$
Therefore, $b_n \rightarrow \ k, \ where \ k \in [b, \infty)$

> **helmut wrote:**
>
> Step 3:
> Show that b is the least upper bound of A.

Let $A = \{a_n | n \in \mathbb{N}\}$
Let $B = [b, \infty)$

Have to show:
$sup(A) = b \iff b \geq a, \ \forall \ a \in A, \ and \ b \leq x, \ \forall \ x \in B$
$\iff$
$b = UB(A), \ and \ b \leq x, \ \forall \ x \in B$

Let $m < b$

Suppose
$sup(A) = m \iff$
$m \geq a, \ \forall \ a \in A, \ and \ m \leq x, \ \forall \ x \in B \iff$
$m = UB(A), \ and \ m \leq x, \ \forall \ x \in B$

However, if $m < b \Rightarrow m \notin B \Rightarrow m \neq UB(A) \Rightarrow m \neq sup(A)$
Additionally, earlier on we showed $\forall a \in A, \ a \leq \ b < b + \delta \ \forall \delta > 0$
Therefore, $b \geq a, \ \forall \ a \in A, \ and \ b \leq x, \ \forall \ x \in B \iff \ b = sup(A)$

Questions?

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helmut | **Post subject:** Re: 2.19 | **Posted:** Sun, 19 Apr 2020 21:22

online

Site Admin

Joined: Sat, 26 Apr 2003 15:14
Posts: 2259
Location: El Paso TX (USA)

> **Quote:**
> Other half $(CA \Rightarrow \ \ sup \ \ \text{property})$
>
> Let a_n be an increasing, bounded sequence.

Don't you have to start with a bounded set instead of a sequence?

I don't understand your Step 1 either.

____

The greater danger for most of us lies not in setting our aim too high and falling short; but in setting our aim too low, and achieving our mark. - Michelangelo Buonarroti

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solivas11

offline

Member

Joined: Mon, 30 Mar 2020 21:12
Posts: 15

Post subject: Re: 2.19 **Posted:** Wed, 29 Apr 2020 19:15

> **helmut wrote:**
>
> > **Quote:**
> > Other half $(CA \Rightarrow \ \ sup \ \ \text{property})$
> >
> > Let a_n be an increasing, bounded sequence.
>
> Don't you have to start with a bounded set instead of a sequence?

Okay. I've also changed the hints just a bit.

Other half $(CA \Rightarrow \ \ sup\ property)$

Let A be a set bounded from above.
Let B be the set of upper bounds for A i.e.
$B = \{b \in \mathbb{R}| \ b = UB(A) \ is \ true\}$
$\iff \ \ B = \{b \in \mathbb{R}| \ \forall \ a \in A: \ \ a \le b\}$

> **helmut wrote:**
>
> Step 1:
> Show that B is an interval of the form (c, ∞) or of the form $[c, \infty)$ for some $c \in \mathbb{R}$.
> You may use the CA for decreasing sequences.

Choose $a_1 \in A,\ b_1 \in B.$
Now, select $c_1 \in (a_1, b_1) \cap (A \cup B)\ s.t.\ |c_1 - a_1| \leq \frac{1}{2}|b_1 - a_1|$
If $c_1 \in A,\ then\ let\ a_2 = c_1,\ and\ b_2 = b_1.$
If $c_1 \in B,\ then\ let\ a_2 = a_1,\ and\ b_2 = c_1.$
Now, select $c_2 \in (a_2, b_2) \cap (A \cup B)\ s.t.\ |c_2 - a_2| \leq \frac{1}{2}|b_2 - a_2|$
If $c_2 \in A,\ then\ let\ a_3 = c_2,\ and\ b_3 = b_2.$
If $c_2 \in B,\ then\ let\ a_3 = a_2,\ and\ b_3 = c_2.$
.
.
.
Now, select $c_m \in (a_m, b_m) \cap (A \cup B)\ s.t.\ |c_m - a_m| \leq \frac{1}{2}|b_m - a_m|$
If $c_m \in A,\ then\ let\ a_{m+1} = c_m,\ and\ b_{m+1} = b_m.$
If $c_m \in B,\ then\ let\ a_{m+1} = a_m,\ and\ b_{m+1} = c_m.$

By repeating the above, an increasing sequence a_n and a decreasing sequence b_n are created.
Define the range of a_n as $\{a_n | n \in \mathbb{N}\}$, and define the range of b_n as $\{b_n | n \in \mathbb{N}\}$
Since a_n is increasing and bounded from above, via CA $a_n \rightarrow c \in \mathbb{R}$.
Since b_n is decreasing and bounded from below, via CA $b_n \rightarrow c \in \mathbb{R}$

Since $b_n \rightarrow c$, then either $c \in \{b_n | n \in\ \mathbb{N}\}$ or $c \notin \{b_n | n \in \mathbb{N}\}$,
and since B is unbounded from above, then B is in the form of $[c, \infty)$ or (c, ∞).

> **helmut wrote:**
>
> Step 2:
> Show $B = [c, \infty)$. How: It suffices to show that the limit of any decreasing sequence of elements in B converges to an element in B. (Why?)

Suppose $c \notin B \iff c \neq UB(A) \iff \exists n \in \mathbb{N}\ s.t.\ a_n > c$
However, since $a_n \rightarrow c$, and a_n is increasing and bounded, then $a_n \leq c\ \forall n \in \mathbb{N}$, otherwise would contradict increasing.
Thus, B is of the form $[c, \infty)$

> **helmut wrote:**
>
> Step 3:
> Show that c is the least upper bound of A.

Have to show:
$sup(A) = c \iff c \geq a,\ \forall\ a \in A,\ and\ c \leq x,\ \forall\ x \in B$
$\iff$
$c = UB(A),\ and\ c \leq x,\ \forall\ x \in B$

Let $m < c$

Suppose

$$sup(A) = m \iff$$
$$m \geq a,\ \forall\ a \in A,\ and\ m \leq x,\ \forall\ x \in B \iff$$
$$m = UB(A),\ and\ m \leq x,\ \forall\ x \in B$$

However, if $m < c \Rightarrow m \notin B \Rightarrow m \neq UB(A) \Rightarrow m \neq sup(A)$
Therefore, $b \geq a,\ \forall\ a \in A,\ and\ b \leq x,\ \forall\ x \in B \iff b = sup(A)$

Hope these fixes were sufficient.

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helmut

online

Site Admin

Joined: Sat, 26 Apr 2003 15:14
Posts: 2259
Location: El Paso TX (USA)

Post subject: Re: 2.19

Posted: Thu, 30 Apr 2020 15:58

In Step 1, it looks like one could always pick $c_n = a_1$. You don't want that, right?

Also: Why is $(a_1, b_1) \cap (A \cup B) \neq \emptyset$?

____

The greater danger for most of us lies not in setting our aim too high and falling short; but in setting our aim too low, and achieving our mark. - Michelangelo Buonarroti

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solivas11

offline

Member

Joined: Mon, 30 Mar 2020 21:12
Posts: 15

Post subject: Re: 2.19

Posted: Thu, 30 Apr 2020 17:33

> **helmut wrote:**
> In Step 1, it looks like one could always pick $c_n = a_1$. You don't want that, right?

Since I set it up as an open interval, I thought I would avoid $c_n = a_1$. I did think about letting $c_1 = \frac{a_1+b_1}{2}$ - I could incorporate an algorithm like that.

> **helmut wrote:**
> Also: Why is $(a_1, b_1) \cap (A \cup B) \neq \emptyset$?

If A is finite, and if $a_1 \neq max(A) \Rightarrow \exists w \in A\ s.t. w > a_1$.
If A is infinite, then $\exists w \in A\ s.t. w > a_1$, assuming $a_1 \neq sup(A)$, but it hasn't been demonstrated what the sup is.

B is an infinite set in the $\mathbb{R}$. Unless $b_1 = inf(B)$, then $\exists w \in B\ s.t. w < b_1$.

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solivas11

offline

Member

Joined: Mon, 30 Mar 2020 21:12
Posts: 15

Post subject: Re: 2.19

Posted: Thu, 30 Apr 2020 18:57

Updated.

Other half ($CA \Rightarrow \quad sup\ property$)

Let A be a set bounded from above.
Let B be the set of upper bounds for A i.e.
$B = \{b \in \mathbb{R} |\ b = UB(A)\ is\ true\}$
$\iff\ B = \{b \in \mathbb{R} |\ \forall\ a \in A:\ a \leq b\}$

> **helmut wrote:**
>
> Step 1:
> Show that B is an interval of the form (c, ∞) or of the form $[c, \infty)$ for some $c \in \mathbb{R}$. You may use the CA for decreasing sequences.

Choose $a_1 \in A,\ b_1 \in B.$

Now, select $c_1 = \dfrac{a_1 + b_1}{2}$
If $c_1 \in B,\ then\ let\ a_2 = a_1,\ and\ b_2 = c_1.$
If $c_1 \notin B,\ \Rightarrow c_1 \neq UB(A)\ \Rightarrow \exists a_2 \in A\ s.t.\ a_2 > c_1.\ Also\ let\ b_2 = b_1.$

Now, select $c_2 = \dfrac{a_2 + b_2}{2}$
If $c_2 \in B,\ then\ let\ a_3 = a_2,\ and\ b_3 = c_2.$
If $c_2 \notin B,\ \Rightarrow c_2 \neq UB(A)\ \Rightarrow \exists a_3 \in A\ s.t.\ a_3 > c_2.\ Also\ let\ b_3 = b_2.$

.

.

.
Now, select $c_m = \dfrac{a_m + b_m}{2}$
If $c_m \in B,\ then\ let\ a_{m+1} = a_m,\ and\ b_{m+1} = c_m.$
If
$c_m \notin B,\ \Rightarrow c_m \neq UB(A)\ \Rightarrow \exists a_{m+1} \in A\ s.t.\ a_{m+1} > c_m.\ Let\ b_{m+1} = b_m.$

By repeating the above, an increasing sequence a_n and a decreasing sequence b_n are created.
Define the range of a_n as $\{a_n | n \in \mathbb{N}\}$, and define the range of b_n as $\{b_n | n \in \mathbb{N}\}$
Since a_n is increasing and bounded from above, via CA $a_n \to c \in \mathbb{R}$.
Since b_n is decreasing and bounded from below, via CA $b_n \to c \in \mathbb{R}$

Since $b_n \to c$, then either $c \in \{b_n | n \in\ \mathbb{N}\}$ or $c \notin \{b_n | n \in \mathbb{N}\}$,
and since B is unbounded from above, then B is in the form of $[c, \infty)$ or (c, ∞).

> **helmut wrote:**
>
> Step 2:
> Show $B = [c, \infty)$. How: It suffices to show that the limit of any decreasing sequence of elements in B converges to an element in B. (Why?)

Suppose $c \notin B \iff c \neq UB(A) \iff \exists n \in \mathbb{N}\ s.t.\ a_n > c$
However, since $a_n \to c$, and a_n is increasing and bounded, then $a_n \leq c\ \forall n \in \mathbb{N}$, otherwise would contradict increasing.
Thus, B is of the form $[c, \infty)$

> **helmut wrote:**
>
> Step 3:
> Show that c is the least upper bound of A.

Have to show:
$sup(A) = c \iff c \geq a,\ \forall\ a \in A,\ and\ c \leq x,\ \forall\ x \in B$
$\iff$
$c = UB(A),\ and\ c \leq x,\ \forall\ x \in B$

Let $m < c$

Suppose
$sup(A) = m \iff$
$m \geq a,\ \forall\ a \in A,\ and\ m \leq x,\ \forall\ x \in B \iff$
$m = UB(A),\ and\ m \leq x,\ \forall\ x \in B$

However, if $m < c \Rightarrow m \notin B \Rightarrow m \neq UB(A) \Rightarrow m \neq sup(A)$
Therefore, $b \geq a,\ \forall\ a \in A,\ and\ b \leq x,\ \forall\ x \in B \iff b = sup(A)$

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helmut

online

Site Admin

Joined: Sat, 26 Apr 2003 15:14
Posts: 2259
Location: El Paso TX (USA)

Post subject: Re: 2.19

Posted: Sun, 03 May 2020 12:24

Very nice, Sergio! 1 hard-earned credit!

The greater danger for most of us lies not in setting our aim too high and falling short; but in setting our aim too low, and achieving our mark. - Michelangelo Buonarroti

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