Preface

Inquiry Based Learning. These notes are designed for classes using Inquiry Based Learning, pioneered—among others—by the eminent mathematician Robert L. Moore, who taught at the University of Texas at Austin from 1920–69. The basic idea behind this method is that you can only learn how to do mathematics by doing mathematics. Here are two famous quotes attributed to R.L. Moore:

> *"There is only one math book, and this book has only one page with a single sentence: Do what you can!'"*

> *"That Student is Taught the Best Who is Told the Least."*

Prerequisites. The only prerequisites for this course consist of knowledge of Calculus and familiarity with set notation and the Method of Proof. These prerequisites can be found in any Calculus book, and, for example, in inquiry based learning textbooks by Carol Schumacher [25] or Margie Hale [9].

Ground Rules. Expect this course to be quite different from other mathematics courses you have taken. Here are the ground rules we will be operating under:

- These notes contain "exercises" and "tasks". You will solve these problems at home and then present the solutions in class. I will call on students at random to present "exercises"; I will call on volunteers to present solutions to the "tasks".
- When you are in the audience, you are expected to be actively engaged in the presentation. This means checking to see if every step of the presentation is clear and convincing to you, and speaking up when it is not. When there are gaps in the reasoning, the students in class will work together to fill the gaps.
- I will only serve as a moderator. My major contribution in class will consist of asking guiding and probing questions. I will also occasionally give short presentations to put topics into a wider context, or to briefly talk about additional concepts not dealt with in the notes.
- You may use only these notes and your own class notes; you are not allowed to consult other books or materials. You must not talk to anyone outside of class about the assignments. You are encouraged to collaborate with other class participants; if you do, you must acknowledge their contribution during your presentation. Exemptions from these restrictions require prior approval by the instructor.

- Your instructor is an important resource for you. I expect frequent visits from all of you during my office hours—many more visits than in a "normal" class. Among other things, you probably will want to come to my office to ask questions about concepts and assigned problems, you will probably occasionally want to show me your work before presenting it in class, and you probably will have times when you just want to talk about the frustrations you may experience.

- It is of paramount importance that we all agree to create a class atmosphere that is supportive and non-threatening to all participants. Disparaging remarks will be tolerated neither from students nor from the instructor.

Historical Perspective. This course gives an "Introduction to Analysis". After its discovery, Calculus turned out to be extremely useful in solving problems in Physics. Ad hoc justifications were used by the generation of mathematicians following Isaac Newton (1643–1727) and Gottfried Wilhelm von Leibniz (1646–1716), and even later by mathematicians such as Leonhard Euler (1707–1783), Joseph-Louis Lagrange (1736–1813) and Pierre-Simon Laplace (1749–1827). A mathematical argument given by Euler, for example, did not differ much from the kind of "explanations" you have seen in your Calculus classes.

In the first third of the nineteenth century, in particular with the publication of the essay *Théorie analytique de la chaleur* by Jean Baptiste Joseph Fourier (1768–1830) in 1822, fundamental problems with this approach of doing mathematics arose: The leading mathematicians in Europe just did not know when Fourier's ingenious method of approximating functions by trigonometric series worked, and when it failed! This led to the quest for putting the concepts of Calculus on a sound foundational basis: *What exactly does it mean for a sequence to converge? What exactly is a function? What does it mean for a function to be differentiable? When is the integral of an infinite sum of functions equal to the infinite sum of the integrals of the functions?* Etc, etc.[1]

As these fundamental questions were investigated and consequently answered by mathematicians such as Augustin Louis Cauchy (1789–1857), Bernhard Bolzano (1781–1848), G. F. Bernhard Riemann (1826–1866), Karl Weierstraß (1815–1897), and many others, the word "Analysis" became the customary term to describe this kind of "Rigorous Calculus". The progress in Analysis during the latter part of the nineteenth century and the rapid progress in the twentieth century would not have been possible without this revitalization of Calculus.

[1] Eventually the "crisis" in Analysis also led to renewed interest into general questions concerning the nature of Mathematics. The resulting work of mathematicians and logicians such as Gottlob Frege (1848–1925), Richard Dedekind (1831–1916), Georg Cantor (1845–1918), and Bertrand Russell (1872–1970), David Hilbert (1862–1943), Kurt Gödel (1906–1978) and Paul Cohen (1934–), Luitzen Egbertus Jan Brouwer (1881–1966) and Arend Heyting (1898–1980) has fundamentally impacted all branches of mathematics and its practitioners. For a fascinating description and a very readable account of these developments and how they led to the theory of computing, see [4].

Consequently, in this course we will investigate (or in many cases revisit) the fundamental concepts in single-variable Calculus: Sequences and their convergence behavior, local and global consequences of continuity, properties of differentiable functions, integrability, and the relation between differentiability and integrability.

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The material presented has been class-tested with students at the University of Texas at El Paso for several years. My students have patiently suffered through earlier versions of this book and have improved the presentation in many ways through their helpful suggestions and comments.

Finally I am grateful to Michael Starbird who exposed me to the modified Moore Method as a graduate student. Among all the graduate classes I took I still remember the material in his topology class the best.

1 Introduction

When we want to study a subject in Mathematics, we first have to agree upon what we assume we all already understand.

In this course we will assume that we are familiar with the Real Numbers, in the sequel denoted by $\mathbb{R}$. Before we list the basic axioms the Real Numbers satisfy, we will briefly review more elementary concepts of numbers.

1.1 The Set of Natural Numbers

When we start learning Mathematics in elementary school, we live in the world of NATURAL NUMBERS, which we will denote by $\mathbb{N}$:

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$

Natural numbers are the "natural" objects to count things around us with. The first thing we learn is to add natural numbers, then later on we start to multiply.

Besides their existence, we will take the following characterization of the Natural Numbers $\mathbb{N}$ for granted throughout the course:

Axiom N1 $1 \in \mathbb{N}$.

Axiom N2 If $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$.

Axiom N3 If $n \neq m$, then $n + 1 \neq m + 1$.

Axiom N4 There is no natural number $n \in \mathbb{N}$, such that $n + 1 = 1$.

Axiom N5 If a subset $M \subseteq \mathbb{N}$ satisfies (1) $1 \in M$, and (2) $m \in M \Rightarrow m + 1 \in M$, then $M = \mathbb{N}$.

The first four axioms describe the features of the counting process: We start counting at 1, every counting number has a "successor", and counting is not "cyclic". The last axiom guarantees the **Principle of Induction**:

Task 1.1
Let $P(n)$ be a predicate with domain $\mathbb{N}$. If

1. $P(1)$ is true, and
2. Whenever $P(n)$ is true, then $P(n + 1)$ is true,

then $P(n)$ is true for all $n \in \mathbb{N}$.

1.2 Integers, Rational and Irrational Numbers

Deficiencies of the system of natural numbers start to appear when we want to divide—the quotient of two natural numbers is not necessarily a natural number, or when we want to subtract—the difference of two natural numbers is not necessarily a natural number. This leads quite naturally to two extensions of the concept of number.

The set of INTEGERS, denoted by $\mathbb{Z}$, is the set

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}.$$

The set of RATIONAL NUMBERS $\mathbb{Q}$ is defined as

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}.$$

Real numbers that are not rational are called IRRATIONAL NUMBERS. The existence of irrational numbers, first discovered by the Pythagoreans in about 520 B.C., must have come as a major surprise to Greek Mathematicians:

Task 1.2
Show that the square root of 2 is irrational. ($\sqrt{2}$ is the positive real number whose square is 2.)

1.3 Groups

Next we will put the properties of numbers and their behavior with respect to the standard arithmetic operations into a wider context by introducing the concept of an "abelian group" and, in the next section, the concept of a "field".

A set G with a binary operation $*$ is called an ABELIAN GROUP[2], if $(G, *)$ satisfies the following axioms:

G1 $*$ is a map from $G \times G$ to G.

G2 (Associativity) For all $a, b, c \in G$

$$(a * b) * c = a * (b * c)$$

G3 (Commutativity) For all $a, b \in G$

$$a * b = b * a$$

[2]Named in honor of Niels Henrik Abel (1802–1829).

G4 (Existence of a neutral element) There is an element $n \in G$, called the neutral element of G, such that for all $a \in G$

$$a * n = a$$

G5 (Existence of inverse elements) For every $a \in G$ there exists $b \in G$, called the inverse of a, such that

$$a * b = n$$

The sets $\mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ are examples of abelian groups when endowed with the usual addition $+$. The neutral element in these cases is 0; it is customary to denote the inverse element of a by $-a$.

The sets $\mathbb{Q} \setminus \{0\} = \{r \in \mathbb{Q} \mid r \neq 0\}$ and $\mathbb{R} \setminus \{0\}$ also form abelian groups under the usual multiplication $\cdot$. In these cases we denote the neutral element by 1; the inverse element of a is customarily denoted by $1/a$ or by a^{-1}.

Exercise 1.3
Write down the axioms **G1–G5** explicitly for the set $\mathbb{Q} \setminus \{0\}$ with the binary operation $\cdot$ (i.e., multiplication).

Addition and multiplication of rational and real numbers interact in a reasonable manner—the following DISTRIBUTIVE LAW holds:

DL For all $a, b, c \in \mathbb{R}$

$$(a + b) \cdot c = (a \cdot c) + (b \cdot c)$$

1.4 Fields

In short, a set F together with an addition $+$ and a multiplication $\cdot$ is called a FIELD, if

F1 $(F, +)$ is an abelian group (with neutral element 0).
F2 $(F \setminus \{0\}, \cdot)$ is an abelian group (with neutral element 1).
F3 For all $a, b, c \in F$: $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$.

The set of rational numbers and the set of real numbers are examples of fields.

Another example of a field is the set of complex numbers $\mathbb{C}$:

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

Addition and multiplication of complex numbers are defined as follows:

$$(a+bi)+(c+di)=(a+c)+(b+d)i,$$

and

$$(a+bi)\cdot(c+di)=(ac-bd)+(ad+bc)i,$$

respectively.

A field F endowed with a relation $\leq$ is called an ORDERED FIELD if

O1 (Antisymmetry) For all $x, y \in F$

$$x \leq y \text{ and } y \leq x \text{ implies } x = y$$

O2 (Transitivity) For all $x, y, z \in F$

$$x \leq y \text{ and } y \leq z \text{ implies } x \leq z$$

O3 For all $x, y \in F$

$$x \leq y \text{ or } y \leq x$$

O4 For all $x, y, z \in F$

$$x \leq y \text{ implies } x + z \leq y + z$$

O5 For all $x, y \in F$ and all $0 \leq z$

$$x \leq y \text{ implies } x \cdot z \leq y \cdot z$$

If $x \leq y$ and $x \neq y$, we write $x < y$. Instead of $x \leq y$, we also write $y \geq x$.

Both the rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$ form ordered fields. The complex numbers $\mathbb{C}$ cannot be ordered in such a way.

1.5 The Completeness Axiom

You probably have seen books entitled “Real Analysis” and “Complex Analysis” in the library. There are no books on “Rational Analysis”.

Why? What is the main difference between the two ordered fields of $\mathbb{Q}$ and $\mathbb{R}$?—The ordered field $\mathbb{R}$ of real numbers is COMPLETE: sequences of real numbers have the following property.

C Let (a_n) be an increasing sequence of real numbers. If (a_n) is bounded from above, then (a_n) converges.

The ordered field $\mathbb{Q}$ of rational numbers, on the other hand, is **not** complete. It should therefore not surprise you that the Completeness Axiom will play a central part throughout the course! We will discuss this axiom in great detail in Section 2.3.

The complex numbers $\mathbb{C}$ also form a complete field. Section 2.6 will give an idea how to write down an appropriate completeness axiom for the field $\mathbb{C}$.

1.6 Summary: An Axiomatic System for the Set of Real Numbers

Below is a summary of the properties of the real numbers $\mathbb{R}$ we will take for granted throughout the course:

The set of real numbers $\mathbb{R}$ with its natural operations of $+, \cdot$, and $\leq$ forms a complete ordered field. This means that the real numbers satisfy the following axioms:

Axiom 1 $+$ is a map from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$.

Axiom 2 For all $a, b, c \in \mathbb{R}$

$$(a + b) + c = a + (b + c)$$

Axiom 3 For all $a, b \in \mathbb{R}$

$$a + b = b + a$$

Axiom 4 There is an element $0 \in \mathbb{R}$ such that for all $a \in \mathbb{R}$

$$a + 0 = a$$

Axiom 5 For every $a \in \mathbb{R}$ there exists $b \in \mathbb{R}$ such that

$$a + b = 0$$

Axiom 6 $\cdot$ is a map from $\mathbb{R} \setminus \{0\} \times \mathbb{R} \setminus \{0\}$ to $\mathbb{R} \setminus \{0\}$.

Axiom 7 For all $a, b, c \in \mathbb{R} \setminus \{0\}$

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

Axiom 8 For all $a, b \in \mathbb{R} \setminus \{0\}$

$$a \cdot b = b \cdot a$$

Axiom 9 There is an element $1 \in \mathbb{R} \setminus \{0\}$ such that for all $a \in \mathbb{R} \setminus \{0\}$

$$a \cdot 1 = a$$

Axiom 10 For every $a \in \mathbb{R} \setminus \{0\}$ there exists $b \in \mathbb{R} \setminus \{0\}$ such that

$$a \cdot b = 1$$

Axiom 11 For all $a, b, c \in \mathbb{R}$

$$(a + b) \cdot c = (a \cdot c) + (b \cdot c)$$

Axiom 12 For all $a, b \in \mathbb{R}$

$$a \leq b \text{ and } b \leq a \text{ implies } a = b$$

Axiom 13 For all $a, b, c \in \mathbb{R}$

$$a \leq b \text{ and } b \leq c \text{ implies } a \leq c$$

Axiom 14 For all $a, b \in \mathbb{R}$

$$a \leq b \text{ or } a \geq b$$

Axiom 15 For all $a, b, c \in \mathbb{R}$

$$a \leq b \text{ implies } a + c \leq b + c$$

Axiom 16 For all $a, b \in \mathbb{R}$ and all $c \geq 0$

$$a \leq b \text{ implies } a \cdot c \leq b \cdot c$$

Axiom 17 Let (a_n) be an increasing sequence of real numbers. If (a_n) is bounded from above, then (a_n) converges.

1.7 Maximum and Minimum

Given a non-empty set A of real numbers, a real number b is called MAXIMUM OF THE SET A, if $b \in A$ and $b \geq a$ for all $a \in A$. Similarly, a real number s is called MINIMUM OF THE SET A, if $s \in A$ and $s \leq a$ for all $a \in A$. We write $b = \max A$, and $s = \min A$.

For example, the set $\{1, 3, 2, 0, -7, \pi\}$ has minimum -7 and maximum π, the set of natural numbers $\mathbb{N}$ has 1 as its minimum, but fails to have a maximum.

Exercise 1.4

Show that a set can have at most one maximum.

Exercise 1.5
Characterize all subsets A of the set of real numbers with the property that $\min A = \max A$.

Task 1.6
Show that finite non-empty sets of real numbers always have a minimum.

1.8 The Absolute Value

The ABSOLUTE VALUE of a real number a is defined as

$$|a| = \max\{a, -a\}.$$

For instance, $|4| = 4$, $|-\pi| = \pi$. Note that the inequalities $a \leq |a|$ and $-a \leq |a|$ hold for all real numbers a.

The quantity $|a - b|$ measures the distance between the two real numbers a and b on the real number line; in particular $|a|$ measures the distance of a from 0.

The following result is known as the **triangle inequality**:

Exercise 1.7
For all $a, b \in \mathbb{R}$:

$$|a + b| \leq |a| + |b|$$

A related result is called the **reverse triangle inequality**:

Exercise 1.8
For all $a, b \in \mathbb{R}$:

$$|a - b| \geq \Big| |a| - |b| \Big|$$

You will use both of these inequalities frequently throughout the course.

1.9 Natural Numbers and Dense Sets inside the Real Numbers

In the sequel, we will also assume the following axiom for the Natural Numbers, even though it can be deduced from the Completeness Axiom of the Real Numbers (see Optional Task 2.1):

> **Axiom N6** For every positive real number $s \in \mathbb{R}$, $s > 0$, there is a natural number $n \in \mathbb{N}$ such that $n - 1 \leq s < n$.

Exercise 1.9
Show that for every positive real number r, there is a natural number n, such that $0 < \frac{1}{n} < r$.

We say that a set A of real numbers is DENSE in $\mathbb{R}$, if for all real numbers $x < y$ there is an element $a \in A$ satisfying $x < a < y$.

Task 1.10
The set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$.

Task 1.11
The set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.